TESTS OF SYMMETRY BASED ON THE CRAMÉR-VON MISES AND WATSON STATISTICS

By

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Three Cramér-von Mises type statistics for testing symmetry about zero have recently appeared in the literature. Two of these have the desirable property of invariance with respect to taking the negatives of the observations (Invariance Property I). However, none of these statistics possesses the property of invariance with respect to taking the reciprocals of the observations (Invariance Property II). In this dissertation, several related statistics, which possess both invariance properties, are developed.

In Chapter 2, two classes of statistics obtained by modifying the two-sample Cramér-von Mises statistic are studied. Statistics in these two classes are known to lead to consistent tests of symmetry and to possess Invariance Property I. In addition, a relationship between the two classes is established, and each class is shown to contain one of the above existing statistics. By combining statistics from the two classes, a third class is obtained. Statistics in the third class are shown to lead to consistent tests of symmetry and to

possess Invariance Property II. Exact critical values are given for one of the statistics in the third class.

Since the asymptotic null distribution of statistics in the third class is not available at this time, a class of statistics obtained from a modification of the Watson two-sample statistic is defined in Chapter 3. Statistics in this fourth class are shown to lead to consistent tests of symmetry and to possess Invariance Property I. In addition, one of the statistics in the fourth class is shown to possess both invariance properties, and exact critical values are calculated for this statistic. The common asymptotic null distribution for statistics in the fourth class is shown to be the asymptotic null distribution of the Cramér-von Mises goodness-of-fit and two-sample statistics.

CHAPTER 1

INTRODUCTION

1.1 Literature Review

This dissertation explores the problem of testing whether a random variable with a continuous cumulative distribution function (CDF) is distributed symmetrically about a known point. A random variable with a continuous CDF, F, will be said to have a symmetric distribution about the point θ, if, for all y,

$$F(\theta - y) = 1 - F(\theta + y).$$

Without loss of generality, the value of θ can be taken as zero, since $X-\theta$ has a symmetric distribution about zero if and only if X has a symmetric distribution about θ. Hence, the hypothesis of symmetry can be defined as follows:

Definition 1.1.1:

A CDF, F, satisfies the hypothesis of symmetry, H_o, provided $F \in \Phi_s$, where Φ_s denotes the set of all continuous CDFs, such that for all y,

$$F(-y) = 1 - F(y). \tag{1.1.1}$$

Note that if Φ denotes the set of all continuous CDFs, then $\Phi - \Phi_s$ is the set of all alternatives to H_o.

A variety of statistics are available for testing the hypothesis

$$H_o: \quad F(y) = G(y) \text{ for all } y,$$

where F and G denote the unknown CDFs of two populations. These statistics will be referred to as two-sample statistics. Several two-sample statistics involve the empirical (sample) distribution function (EDF), $F_n(\cdot)$.[1] One such statistic is the two-sample Cramér-von Mises statistic, defined by Kiefer [10] as

$$\omega_n^2 = n \int_{-\infty}^{+\infty} [F_n(y) - G_n(y)]^2 \, dH_n(y),$$

where F_n (G_n) denotes the EDF based on a sample of size n from the population with CDF, F (G), and H_n denotes the EDF based on the combined sample. Another two-sample EDF statistic, due to Watson [20], is defined by

$$\mu_n^2 = n \int_{-\infty}^{+\infty} [F_n(y) - G_n(y) - \int_{-\infty}^{+\infty} (F_n(w) - G_n(w)) \, dH_n(w)]^2 \, dH_n(y),$$

where F_n, G_n and H_n are as defined above. It should be noted here that similar but slightly more complicated forms of ω_n^2 and μ_n^2 exist, for use in the case of unequal sample sizes.

Recently, several articles have appeared in the literature in which a statistic for testing symmetry about zero is obtained by modifying one of the two-sample EDF statistics ([3], [4], [11], [12], [15] and [17]). Such modifications of ω_n^2 and μ_n^2 will be studied in this dissertation.

To date, three symmetry statistics obtained from ω_n^2 have been studied. Orlov [12] and Rothman and Woodroofe [15] have independently

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[1] If $X_1, X_2, \ldots, X_n$ denotes a random sample, the EDF $F_n(\cdot)$, is defined by

$$F_n(y) = n^{-1}(\text{the number of } X_i\text{s} \leq y).$$

proposed essentially equivalent statistics. Orlov's statistic is obtained from ω_n^2 by replacing $[F_n(y) - G_n(y)]$ with the expression, $[F_n(y) + F_n(-y) - 1]$, and is given by

$$L(0) = \frac{n}{2}\int_{-\infty}^{0} [F_n(y) + F_n(-y) - 1]^2 \, dF_n(y).$$

Orlov points out that the value of L(0) depends only on the values of $[F_n(y) + F_n(-y) - 1]^2$ at its discontinuity points, $X_1, X_2, \ldots, X_n$, and that these values may be taken to be anywhere between the right-hand and left-hand limits of $[F_n(y) + F_n(-y) - 1]^2$ without altering the asymptotic properties of L(0). Orlov derives the asymptotic null distribution of L(0) and gives a table of percentage points for this distribution.

Rothman and Woodroofe [15] have chosen a specific modification of $[F_n(y) + F_n(-y) - 1]$, and their statistic, R_n', is defined as follows:

$$R_n' = n\int_{-\infty}^{+\infty} [F_n'(y) + F_n'(-y) - 1]^2 \, dF_n(y), \qquad (1.1.2)$$

where

$$F_n'(y) = \frac{1}{2}[F_n(y+) + F_n(y-)]. \qquad (1.1.3)$$

This modification of the EDF has the desirable effect of producing a statistic which is invariant with respect to the transformation $X_j \to -X_j$. In other words, if $\underline{X}$ and $-\underline{X}$ denote, respectively, the vectors $(X_1, X_2, \ldots, X_n)$ and $(-X_1, -X_2, \ldots, -X_n)$, where $X_1, X_2, \ldots, X_n$ denotes a random sample from $F \in \Phi$, then

$$R_n'(\underline{X}) = R_n'(-\underline{X}) \quad \text{w.p. 1},$$

where $R_n'(\underline{X})$ denotes the value of the statistic, R_n', for the sample $\underline{X}$. In the sequel, this property will be called Invariance Property I.

The asymptotic null distribution of R_n', equivalent to that of L(0), is obtained by Rothman and Woodroofe. In addition, they give approximate critical values for selected sample sizes and levels, α, based on a Monte Carlo study. Tests based on R_n' are shown to be consistent against all alternatives in $\Phi - \Phi_s$, and the asymptotic distribution of R_n' under a sequence of local alternatives is also investigated.

The third modification of ω_n^2 is due to Srinivasan and Godio [17]. A discussion of this statistic requires the definition of the following functions, to be used throughout the dissertation.

Definition 1.1.2:

Let $(Z_{(1)}, Z_{(2)},\ldots,Z_{(n)})$ denote the vector obtained by ordering the elements of $\underline{Z} = (Z_1,Z_2,\ldots,Z_n)$ in ascending order according to the order relation given by

$$Z_i \text{ "<" } Z_j \quad \text{if} \quad |Z_i| < |Z_j|.$$

In addition, if P denotes a proposition, let the indicator function $I[\cdot]$ be given by

$$I[P] = \begin{cases} 1 & \text{if P is true} \\ 0 & \text{if P is false.} \end{cases} \tag{1.1.4}$$

Then, the functions $\delta_k(\cdot)$, $P_k(\cdot)$ and $N_k(\cdot)$, for k=1,2,...,n, are defined by

$$\delta_k(\underline{Z}) = \begin{cases} -1 & \text{if } Z_{(k)} < 0 \\ 1 & \text{if } Z_{(k)} \geq 0, \end{cases} \tag{1.1.5}$$

$$P_k(\underline{Z}) = \sum_{j=1}^{k} I[\delta_j(\underline{Z}) = 1], \tag{1.1.6}$$

and

$$N_k(\underline{Z}) = \sum_{j=1}^{k} I[\delta_j(\underline{Z}) = -1]. \tag{1.1.7}$$

If $\underline{Z}$ denotes a random sample from a continuous CDF, then

$$P(|Z_i| = |Z_j|) = P(Z_i = 0) = 0, \quad i \neq j.$$

Thus, with probability one, the functions $\delta_k(\cdot)$, $P_k(\cdot)$ and $N_k(\cdot)$ are well defined.

The statistic proposed by Srinivasan and Godio, S_n', may now be defined as follows:

$$S_n' \equiv S_n'(\underline{X}) = \sum_{k=1}^{n} [N_k(\underline{X}) - \frac{k}{2}]^2 = \frac{1}{4} \sum_{k=1}^{n} [N_k(\underline{X}) - P_k(\underline{X})]^2. \tag{1.1.8}$$

The asymptotic and exact (for n=10(1)20) null distributions of S_n' are given in [17], the former being equivalent to the common asymptotic null distribution of the statistics L(0) and R_n'. In addition, tests based on S_n' are shown to be consistent against all alternatives in $\Phi - \Phi_s$. Although it is not mentioned in [17], it is clear from (1.1.8) that S_n' possesses Invariance Property I.

1.2 Summary of Results

In Chapter 2, two classes of statistics for testing symmetry are defined, one containing R_n', and the other containing a statistic equivalent to S_n'. The statistics in both classes are shown to possess Invariance Property I and to have a common asymptotic null distribution. Tests based on these statistics are shown to be consistent against all alternatives to symmetry about zero. For each statistic in the first class, it is shown that there exists a corresponding statistic in the second class with the same exact null distribution. These pairs of statistics are related further in that the test based on one is equivalent to the test based on the other using the sample of reciprocals, $(X_1^{-1}, X_2^{-1},\ldots,X_n^{-1})$. By combining these pairs, a third class of statistics is defined. Statistics in the third class lead to consistent tests of symmetry, possess Invariance Property I and, in addition, are invariant with respect to the transformation, $X_j \to X_j^{-1}$. In other words, if $\underline{X}^{-1}$ denotes the vector of reciprocals, $(X_1^{-1},X_2^{-1},\ldots,X_n^{-1})$, where $\underline{X}$ denotes a random sample from $F \in \Phi$, and if T denotes a member of the third class of statistics, then

$$T(\underline{X}) = T(\underline{X}^{-1}) \quad \text{w.p. 1.}$$

This invariance property will henceforth be referred to as Invariance Property II.

Exact critical values are given for one member of the third class for n = 10(1)24. However, it does not appear that the common asymptotic null distribution of members of the third class will be easy to obtain. For this reason, a fourth class of statistics, based on a modification of Watson's two-sample statistic, μ_n^2, is defined in

Chapter 3. These statistics are shown to lead to consistent tests of symmetry, as well as to possess Invariance Property I. In addition, the common asymptotic null distribution of statistics in the fourth class is shown to be that of the well-known Cramér-von Mises goodness-of-fit statistic [7]. One member of the fourth class is shown to possess Invariance Property II, and exact critical values for this statistic are given for n = 9(1)20.

Chapter 4 contains a brief description of the problems that are yet to be solved in the general area of the topics considered in this dissertation.

CHAPTER 2

CRAMÉR-VON MISES TYPE STATISTICS FOR TESTING SYMMETRY

2.1 Two Classes of Cramér-von Mises Type Symmetry Statistics

Intuitively, a test of H_o may be based on either of the following:

1. A comparison of estimates of F(y) and (1 - F(-y)) for all values of y.
2. A comparison of estimates of (F(y) - F(0)) and (F(0) - F(-y)) for all values of y.

Since a natural estimator for F(y) is the EDF, $F_n(y)$, a reasonable test statistic can be constructed from any functional of the difference between the estimates to be compared in (1) or (2) above. In (1), this test statistic would be based on

$$F_n(y) - (1 - F_n(-y))$$
$$= F_n(y) + F_n(-y) - 1, \qquad (2.1.1)$$

whereas in (2), it would be based on

$$F_n(y) - F_n(0) - (F_n(0) - F_n(-y))$$
$$= F_n(y) + F_n(-y) - 2F_n(0). \qquad (2.1.2)$$

A modified version of F_n, slightly more general than the one introduced by Rothman and Woodroofe [15], will be used to define the statistics considered in this dissertation. The reason for introducing

this version of the EDF will be noted after the proof of Theorem 2.2.2. The modified EDF is presented in the following definition. Hereafter, $\underline{X} = (X_1, X_2, \ldots, X_n)$ will denote a random sample from $F \in \Phi$, and F_n will denote the standard version of the EDF obtained from $\underline{X}$.

Definition 2.1.1:

Corresponding to each $a \in [0,1]$, the modified EDF, $F_n^{(a)}$, is defined as

$$F_n^{(a)}(y) = \begin{cases} aF_n(y+) + (1-a)F_n(y-) & \text{if } y < 0 \\ \\ (1-a)F_n(y+) + aF_n(y-) & \text{if } y \geq 0, \end{cases} \tag{2.1.3}$$

where $F_n(y+)$ and $F_n(y-)$ are the right-hand and left-hand limits, respectively, of F_n at y. Note that, from (1.1.3),

$$F_n' = F_n^{(\frac{1}{2})}, \tag{2.1.4}$$

and like F_n', $F_n^{(a)}$ differs from F_n only at its discontinuity points, $X_1, X_2, \ldots, X_n$. At these points, $F_n^{(a)}$ lies between the right-hand and left-hand limits of F_n. Specifically, for $i=1,2,\ldots,n$,

$$F_n^{(a)}(X_i) = \begin{cases} F_n(X_i) - \dfrac{(1-a)}{n} & \text{if } X_i < 0 \\ \\ F_n(X_i) - \dfrac{a}{n} & \text{if } X_i \geq 0, \end{cases} \tag{2.1.5}$$

since

$$F_n(X_i) = F_n(X_i+) = F_n(X_i-) + \frac{1}{n}.$$

The following lemma shows that F_n and $F_n^{(a)}$ behave similarly for large samples.

Lemma 2.1.1:

For every $F \in \Phi$ and every $a \in [0,1]$,

$$\sup_y \left| F_n^{(a)}(y) - F(y) \right| \to 0 \quad \text{w.p. 1, as } n \to \infty.$$

Proof:

It is easily seen from (2.1.5) that

$$\sup \left| F_n^{(a)}(y) - F_n(y) \right| \leq \frac{1}{n} \max \{a, (1-a)\} \leq \frac{1}{n}.$$

The desired result now follows from the triangle inequality and the Glivenko-Cantelli Lemma [5].

The following corollary follows directly from (1.1.1) and Lemma 2.1.1. It will be needed in the proof of a later theorem.

Corollary 2.1.1:

Under H_o,

$$\sup_y \left| F_n^{(a)}(y) + F_n^{(a)}(-y) - 1 \right| \to 0 \quad \text{w.p. 1, as } n \to \infty,$$

for every $a \in [0,1]$.

Now, by replacing F_n with the modified EDF, $F_n^{(a)}$, in the comparisons given by (2.1.1) and (2.1.2), and then applying the functional, $\int(\cdot)^2$, two distinct Cramér-von Mises type statistics for testing H_o are produced. Since the index, a, can take any value in [0,1], two classes of statistics actually result. These two classes will be denoted $\left\{R_n^{(a)}: \ 0 \leq a \leq 1\right\}$ and $\left\{S_n^{(a)}: \ 0 \leq a \leq 1\right\}$, and their general members are now defined.

Definition 2.1.2:

For $a \in [0,1]$,

$$R_n^{(a)} \equiv R_n^{(a)}(\underline{X}) = n \int_{-\infty}^{+\infty} [F_n^{(a)}(y) + F_n^{(a)}(-y) - 1]^2 \, dF_n(y), \qquad (2.1.6)$$

and

$$S_n^{(a)} \equiv S_n^{(a)}(\underline{X}) = n \int_{-\infty}^{+\infty} [F_n^{(a)}(y) + F_n^{(a)}(-y) - 2F_n(0)]^2 \, dF_n(y). \qquad (2.1.7)$$

Some properties of $R_n^{(a)}$ and $S_n^{(a)}$ will be considered in the next section. It should be noted here that reasonable tests of H_o are obtained by rejecting H_o for "large" values of $R_n^{(a)}$ or $S_n^{(a)}$. Also, from (2.1.4), $R_n' = R_n^{(\frac{1}{2})}$, so that one member of the class $\{R_n^{(a)}: \ 0 \le a \le 1\}$ has already been studied by Rothman and Woodroofe [15].

2.2 Properties of $R_n^{(a)}$ and $S_n^{(a)}$

As pointed out by Rothman and Woodroofe [15], it is reasonable to require that a test statistic for H_o possess Invariance Property I. In Chapter 1, it was noted that both R_n' and S_n' satisfy this requirement. The next theorem shows that each member of the classes $\left\{R_n^{(a)}: \ 0 \le a \le 1\right\}$ and $\left\{S_n^{(a)}: \ 0 \le a \le 1\right\}$ possesses Invariance Property I.

Theorem 2.2.1:

For every $a \in [0,1]$,

$$R_n^{(a)}(\underline{X}) = R_n^{(a)}(-\underline{X}) \quad \text{w.p. 1}, \qquad (2.2.1)$$

and

$$S_n^{(a)}(\underline{X}) = S_n^{(a)}(-\underline{X}) \quad \text{w.p. 1}. \qquad (2.2.2)$$

Proof:

Let G_n and $G_n^{(a)}$ denote, respectively, the EDF and its modification (according to Definition 2.1.1) for the sample, $-\underline{X}$. If $y < 0$, then

$$\begin{aligned} F_n^{(a)}(y) &= aF_n(y+) + (1-a)F_n(y-) \\ &= n^{-1}\left\{a\sum_{i=1}^{n} I[X_i \le y] + (1-a)\sum_{i=1}^{n} I[X_i < y]\right\} \\ &= n^{-1}\left\{a\sum_{i=1}^{n}\left(1-I[-X_i < -y]\right) + (1-a)\sum_{i=1}^{n}\left(1-I[-X_i \le -y]\right)\right\} \\ &= 1 - aG_n(-y-) - (1-a)G_n(-y+). \end{aligned}$$

Since $y < 0$ implies that $-y > 0$, the last expression equals

$$1 - G_n^{(a)}(-y).$$

Analogously, the same result holds if $y > 0$. Since $F_n(0) = 1 - G_n(0)$ with probability one, for all y,

$$F_n^{(a)}(y) = 1 - G_n^{(a)}(-y) \quad \text{w.p. 1.}$$

Hence, for all y,

$$F_n^{(a)}(y) + F_n^{(a)}(-y) - 1 = -\left[G_n^{(a)}(y) + G_n^{(a)}(-y) - 1\right] \quad \text{w.p. 1,}$$

from which (2.2.1) follows immediately. A sketch of the proof of (2.2.2) is given on page 16, immediately following Corollary 2.2.1.

Recall that in (1.1.8), S_n' was defined in terms of $P_k(\cdot)$, $N_k(\cdot)$ and $\delta_k(\cdot)$, and not in terms of the EDF. The next theorem provides representations of $R_n^{(a)}$ and $S_n^{(a)}$ in terms of $P_k(\cdot)$, $N_k(\cdot)$ and $\delta_k(\cdot)$. These representations not only provide a convenient form for the actual

computation of the test statistics, but also enable one to establish an important relation between $R_n^{(1-a)}$ and $S_n^{(a)}$, $0 \le a \le 1$. In addition, they are useful in proving several theorems which follow.

Theorem 2.2.2:

With probability one, for every $a \in [0,1]$,

$$R_n^{(a)}(\underline{X}) = n^{-2}\sum_{k=1}^{n}\left[N_k(\underline{X}^{-1}) - P_k(\underline{X}^{-1}) + (1-a)\delta_k(\underline{X}^{-1})\right]^2, \qquad (2.2.3)$$

and

$$S_n^{(a)}(\underline{X}) = n^{-2}\sum_{k=1}^{n}\left[N_k(\underline{X}) - P_k(\underline{X}) + (a)\delta_k(\underline{X})\right]^2, \qquad (2.2.4)$$

where $\delta_k(\cdot)$, $P_k(\cdot)$ and $N_k(\cdot)$ are given by (1.1.5), (1.1.6) and (1.1.7).

Proof:

The result given by (2.2.3) will be proved first. If it can be shown that, for $k=1,2,\ldots,n$,

$$F_n^{(a)}(X_{(n-k+1)}) + F_n^{(a)}(-[X_{(n-k+1)}]) - 1$$

$$= n^{-1}\left[N_k(\underline{X}^{-1}) - P_k(\underline{X}^{-1}) + (1-a)\delta_k(\underline{X}^{-1})\right], \qquad (2.2.5)$$

then (2.2.3) will follow immediately from (2.1.6) by summing the squares on both sides of (2.2.5) over the index k, $k=1,2,\ldots,n$.

To establish (2.2.5), expressions involving F_n will be found for P_k and N_k. If $\underline{Y} = \underline{X}^{-1}$, then, from Definition 1.1.2, $(Y_{(n-j+1)})^{-1} = X_{(j)}$.

Hence, for k=1,2,...,n, with probability one,

$$P_k(\underline{X}^{-1}) = \sum_{j=n-k+1}^{n} I\left[X_{(j)} \geq 0\right], \tag{2.2.6}$$

and

$$N_k(\underline{X}^{-1}) = \sum_{j=n-k+1}^{n} I\left[X_{(j)} < 0\right]. \tag{2.2.7}$$

Since $|X_{(i)}| < |X_{(j)}|$ if and only if $i < j$, equivalent expressions of (2.2.6) and (2.2.7) are given by

$$P_k(\underline{X}^{-1}) = \sum_{i=1}^{n} I\left[X_i \geq |X_{(n-k+1)}|\right],$$

and

$$N_k(\underline{X}^{-1}) = \sum_{i=1}^{n} I\left[X_i \leq -|X_{(n-k+1)}|\right].$$

Thus, with probability one, for k=1,2,...,n,

$$P_k(\underline{X}^{-1}) = \begin{cases} n\left[1-F_n(|X_{(n-k+1)}|)\right] & \text{if } X_{(n-k+1)} < 0 \\ n\left[1-F_n(|X_{(n-k+1)}|-)\right] & \text{if } X_{(n-k+1)} \geq 0 \end{cases}$$

$$= n\left\{1-F_n(|X_{(n-k+1)}|) + n^{-1}I\left[X_{(n-k+1)} \geq 0\right]\right\},$$

and

$$N_k(\underline{X}^{-1}) = nF_n(-|X_{(n-k+1)}|).$$

Hence, with probability one,

$$N_k(\underline{X}^{-1}) - P_k(\underline{X}^{-1}) + (1-a)\delta_k(\underline{X}^{-1})$$

$$= n\left\{F_n(-|X_{(n-k+1)}|) - 1 + F_n(|X_{(n-k+1)}|) - n^{-1}I\left[X_{(n-k+1)} \geq 0\right] + n^{-1}(1-a)\delta_{n-k+1}(\underline{X})\right\}$$

$$= n\left\{F_n(X_{(n-k+1)}) + F_n(-X_{(n-k+1)}) - 1 - \alpha_k\right\}, \tag{2.2.8}$$

where

$$\alpha_k = n^{-1}\left\{I\left[X_{(n-k+1)} \geq 0\right] - (1-a)\delta_{n-k+1}(\underline{X})\right\}$$

$$= \begin{cases} \dfrac{(1-a)}{n} & \text{if } X_{(n-k+1)} < 0 \\ \dfrac{a}{n} & \text{if } X_{(n-k+1)} \geq 0 \end{cases} \qquad k = 1,2,\ldots,n.$$

Now, it follows from (2.1.5) that

$$F_n(X_{(n-k+1)}) + F_n(-X_{(n-k+1)}) - \alpha_k = F_n^{(a)}(X_{(n-k+1)}) + F_n^{(a)}(-X_{(n-k+1)}),$$

and, hence, (2.2.5) follows from (2.2.8), which establishes (2.2.3).

The proof of (2.2.4) is almost identical. It can be shown that

$$P_k(\underline{X}) = \sum_{i=1}^{n} I\left[0 \leq X_i \leq X_{(k)}\right],$$

and

$$N_k(\underline{X}) = \sum_{i=1}^{n} I\left[-X_{(k)} \leq X_i < 0\right].$$

This leads to the following equation corresponding to (2.2.5):

$$F_n^{(a)}(X_{(k)}) + F_n^{(a)}(-X_{(k)}) - 2F_n(0)$$

$$= -\frac{1}{n}\left[N_k(\underline{X}) - P_k(\underline{X}) + a\,\delta_k(\underline{X})\right]. \qquad (2.2.9)$$

Thus, (2.2.4) follows as before, and the proof of the theorem is complete.

An immediate consequence of (1.1.8) and (2.2.4) is that

$$S_n^{(0)}(\underline{X}) = (4n^{-2})S_n'(\underline{X}).$$

Hence, Srinivasan and Godio's statistic, S_n', can be expressed in terms of the modified EDF, $F_n^{(a)}$, using (2.1.7), which provides the reason for introducing $F_n^{(a)}$.

The following corollary, which follows directly from Theorem 2.2.2, establishes an important relation between the classes $\left\{R_n^{(a)} : 0 \le a \le 1\right\}$ and $\left\{S_n^{(a)} : 0 \le a \le 1\right\}$.

Corollary 2.2.1:

For every $a \in [0,1]$,

$$R_n^{(1-a)}(\underline{X}^{-1}) = S_n^{(a)}(\underline{X}) \quad \text{w.p. 1.}$$

In other words, the test of symmetry based on $S_n^{(a)}$, using the sample $\underline{X}$, is equivalent to the test based on $R_n^{(1-a)}$, using the sample of reciprocals, $\underline{X}^{-1}$.

As a result of Corollary 2.2.1, the proof of Theorem 2.2.1 can easily be completed by noting that (2.2.2) follows directly from (2.2.1) and Corollary 2.2.1, since the transformations $\underline{X} \to \underline{X}^{-1}$ and $\underline{X} \to -\underline{X}$ are commutative.

In conjunction with Corollary 2.2.1, the following lemma will be useful in establishing several later results.

Lemma 2.2.1:

Let F and F_r denote the CDFs of X and its reciprocal, X^{-1}, respectively. Then,

$$F \in \Phi_s \quad \text{if and only if} \quad F_r \in \Phi_s.$$

Proof:

Let F^* denote the CDF of the random variable -X. Then it follows directly from (1.1.1) that

$$F \in \Phi_s \quad \text{if and only if } F = F^* \text{ and } F \in \Phi. \tag{2.2.10}$$

Also observe that

$$F_r(y) = \begin{cases} F(0) - F(y^{-1}) & \text{if } y < 0 \\ F(0) & \text{if } y = 0 \\ 1 + F(0) - F(y^{-1}) & \text{if } y > 0. \end{cases}$$

Thus, since

$$\lim_{y \to 0-} \left[F(0) - F(y^{-1})\right] = \lim_{y \to 0+} \left[1 + F(0) - F(y^{-1})\right] = F(0)$$

it follows that

$$F = F^* \text{ and } F \in \Phi \quad \text{if and only if} \quad F_r = F_r^* \text{ and } F_r \in \Phi.$$

Combining this result with (2.2.10) completes the proof.

Another consequence of Theorem 2.2.2 is that, for every $a \in [0,1]$, both $R_n^{(a)}$ and $S_n^{(a)}$ are distribution free. Considering $S_n^{(a)}$ first, it is clear from Definition 1.1.2 and Theorem 2.2.2 that $S_n^{(a)}(\underline{X})$ depends on $\underline{X}$ only through

$$\underline{\delta} = (\delta_1(\underline{X}), \delta_2(\underline{X}), \ldots, \delta_n(\underline{X})).$$

A result of Hájek and Sidák ([8], p. 40) shows that, under H_o,

$$P(\underline{\delta} = \underline{d}) = 2^{-n},$$

for every $\underline{d} \in \Delta$, where

$$\Delta = \left\{(d_1, d_2, \ldots, d_n) : d_i = \pm 1\right\}.$$

Hence, for every $a \in [0,1]$, $S_n^{(a)}$ is distribution free under H_o. It now follows from Corollary 2.2.1 and Lemma 2.2.1, that $R_n^{(a)}$ is likewise distribution free under H_o, and that $S_n^{(a)}$ and $R_n^{(1-a)}$ are identically distributed for every $a \in [0,1]$ and every $F \in \Phi_s$. That is, for each

statistic in the class $\left\{ S_n^{(a)} : 0 \le a \le 1 \right\}$, there exists a corresponding statistic in the class $\left\{ R_n^{(a)} : 0 \le a \le 1 \right\}$ with the same null distribution. It should be noted, however, that these pairs of statistics do not produce equivalent tests of H_o, even when n becomes large, since the statistics in a pair will not have equal values for every sample, $\underline{X}$, and since $|R_n^{(1-a)} - S_n^{(a)}|$ does not converge to zero.

Rothman and Woodroofe [15] do not give exact critical values for R_n', and Srinivasan and Godio [17] were not successful in deriving them with the method they employed to obtain the exact values for S_n'. However, from the above discussion, it follows that the critical values given in [17] are also the critical values for $(\frac{1}{4}n^2)\, R_n^{(1)}$. Hence, exact critical values are available for a statistic closely related to $R_n^{(\frac{1}{2})}$. In fact, $R_n^{(1)}$ and $R_n^{(\frac{1}{2})}$ have the same asymptotic null distribution. Since $R_n^{(1)}$ and $S_n^{(0)}$ are identically distributed, this follows from the fact, noted in Section 1.1, that $R_n^{(\frac{1}{2})}$ and $S_n^{(0)}$ have a common asymptotic null distribution. The next theorem shows that this relation between $R_n^{(1)}$ and $R_n^{(\frac{1}{2})}$ holds for any pair of statistics in the R_n class, and for any pair in the S_n class as well.

Theorem 2.2.3:

Under H_o, $R_n^{(a)}$ $(S_n^{(a)})$ and $R_n^{(b)}$ $(S_n^{(b)})$ have the same asymptotic distribution for any a and b in the unit interval. In fact, under H_o,

$$|R_n^{(a)} - R_n^{(b)}| \to 0 \quad \text{w.p. 1, as } n \to \infty,$$

and

$$|S_n^{(a)} - S_n^{(b)}| \to 0 \quad \text{w.p. 1, as } n \to \infty.$$

Proof:

In view of Corollary 2.2.1 and Lemma 2.2.1, it will be sufficient to prove the result only for $R_n^{(a)}$. Observe that

$$R_n^{(a)} = R_n^{(b)} + n \int_{-\infty}^{+\infty} \left\{ \left[F_n^{(a)}(y) + F_n^{(a)}(-y) - F_n^{(b)}(y) - F_n^{(b)}(-y) \right]^2 \right.$$

$$+ 2\left[F_n^{(a)}(y) + F_n^{(a)}(-y) - F_n^{(b)}(y) - F_n^{(b)}(-y) \right]$$

$$\left. \times \left[F_n^{(b)}(y) + F_n^{(b)}(-y) - 1 \right] \right\} dF_n(y),$$

and hence

$$|R_n^{(a)} - R_n^{(b)}| \leq n \left\{ \left[\sup_y |F_n^{(a)}(y) + F_n^{(a)}(-y) - F_n^{(b)}(y) - F_n^{(b)}(-y)| \right]^2 \right.$$

$$+ 2\left[\sup_y |F_n^{(a)}(y) + F_n^{(a)}(-y) - F_n^{(b)}(y) - F_n^{(b)}(-y)| \right]$$

$$\left. \times \left[\sup_y |F_n^{(b)}(y) + F_n^{(b)}(-y) - 1| \right] \right\}.$$

From (2.1.5),

$$\sup_y |F_n^{(a)}(y) + F_n^{(a)}(-y) - F_n^{(b)}(y) - F_n^{(b)}(-y)| \leq 2n^{-1},$$

and thus it follows that

$$|R_n^{(a)} - R_n^{(b)}| \leq o_n(1) + 4 \sup_y |F_n^{(b)}(y) + F_n^{(b)}(-y) - 1|. \qquad (2.2.11)$$

By Corollary 2.1.1, the second term of the last expression, under H_o, converges to zero with probability one, completing the proof of the theorem.

It is now clear, from the fact that $R_n^{(1-a)}$ and $S_n^{(a)}$ are identically distributed, that all statistics in both classes have a common asymptotic null distribution. This agrees with the work of Rothman and Woodroofe [15] and Srinivasan and Godio [17], who independently derived

the same asymptotic distribution for the statistics $R_n^{(\frac{1}{2})}$ and $S_n^{(0)}$, respectively. As noted in Section 1.1, percentage points have been tabulated for this distribution by Orlov [12].

Consistency of the test for symmetry based on any statistic in the classes $\left\{R_n^{(a)} : 0 \le a \le 1\right\}$ or $\left\{S_n^{(a)} : 0 \le a \le 1\right\}$ is now established.

Theorem 2.2.4:

For any $a \in [0,1]$, $R_n^{(a)}$ ($S_n^{(a)}$) is consistent against any alternative in $\Phi - \Phi_s$.

Proof:

Rothman and Woodroofe [15] show that if H_o is not satisfied,

$$R_n^{(\frac{1}{2})} \to +\infty \quad \text{w.p. 1.}$$

Since $\sup_y |F_n^{(\frac{1}{2})}(y) + F_n^{(\frac{1}{2})}(-y) - 1| \le 1$, it follows from (2.2.11) that for any $a \in [0,1]$, $|R_n^{(a)} - R_n^{(\frac{1}{2})}| \le o_n(1) + 4$. Hence for any $F \in \Phi - \Phi_s$,

$$R_n^{(a)} \to +\infty \quad \text{w.p. 1,} \tag{2.2.12}$$

and thus $R_n^{(a)}$ is consistent against any alternative in $\Phi - \Phi_s$ for every $a \in [0,1]$. In view of Corollary 2.2.1 and Lemma 2.2.1, $S_n^{(a)}$ is likewise consistent for every $a \in [0,1]$, completing the proof.

Since exact critical values for $R_n^{(1)}$ are readily available, it is of interest to know whether there is any further advantage to be gained by using $R_n^{(1)}$ instead of $R_n^{(\frac{1}{2})}$. More generally, one might look for a criterion to select a statistic from each of the two classes, $\left\{R_n^{(a)} : 0 \le a \le 1\right\}$ and $\left\{S_n^{(a)} : 0 \le a \le 1\right\}$. Of course, it has been established by Theorem 2.2.3 that the effect of the choice of the index, a, becomes negligible as the sample size becomes large. However, the

value of the index will affect the small sample distributions of both $R_n^{(a)}$ and $S_n^{(a)}$. Specifically, the approximately $2^n\alpha$ sample points (of Δ) comprising the rejection region of an α-level test based on $R_n^{(a)}$ or $S_n^{(a)}$ will depend on the value of the index, a, and hence, so will the small sample power of such a test. The following corollary to Theorem 2.2.2 may be applied to provide a characterization of the effect of the choice of the index, a, on the statistics in the two classes.

Corollary 2.2.2:

With probability one, and for every $a \in [0,1]$,

$$(n^2)S_n^{(a)} = \sum_{k=1}^{n-1} \left[N_k(\underline{X}) - P_k(\underline{X})\right]^2 + (1-a)\left[N_n(\underline{X}) - P_n(\underline{X})\right]^2 - na(1-a), \qquad (2.2.13)$$

and

$$(n^2)R_n^{(a)} = \sum_{k=1}^{n-1} \left[N_k(\underline{X}^{-1}) - P_k(\underline{X}^{-1})\right]^2 + a\left[N_n(\underline{X}^{-1}) - P_n(\underline{X}^{-1})\right]^2 - na(1-a).$$

Proof:

It will be sufficient to prove only the result for $S_n^{(a)}$, as the corresponding result for $R_n^{(1-a)}$ will then follow directly from Corollary 2.2.1. From (2.2.4),

$$n^2S_n^{(a)} = \sum_{k=1}^{n} \left[N_k(\underline{X}) - P_k(\underline{X})\right]^2 + 2a \sum_{k=1}^{n} \left[N_k(\underline{X}) - P_k(\underline{X})\right] \delta_k(\underline{X}) + \sum_{k=1}^{n} \left[a\delta_k(\underline{X})\right]^2, \qquad (2.2.14)$$

and from (1.1.6) and (1.1.7),

$$\left[N_k(\underline{X}) - P_k(\underline{X})\right] = (-1) \sum_{j=1}^{k} \delta_j(\underline{X}). \qquad (2.2.15)$$

Hence, the second term on the right-hand side of (2.2.14) can be rewritten as

$$-2a \sum_{k=1}^{n} \sum_{j=1}^{k} \delta_k(\underline{X})\ \delta_j(\underline{X})$$

$$= -2a \sum_{k=1}^{n} \left[\delta_k(\underline{X})\right]^2 - 2a \sum_{k=1}^{n} \sum_{j=1}^{n} \delta_k(\underline{X})\ \delta_j(\underline{X})$$

$$= -a \sum_{k=1}^{n} \left[\delta_k(\underline{X})\right]^2 - a\left[\sum_{k=1}^{n} \delta_k(\underline{X})\right]^2 .$$

Using (2.2.15) again, this last expression becomes

$$-a \sum_{k=1}^{n} \left[\delta_k(\underline{X})\right]^2 - a\left[N_n(\underline{X}) - P_n(\underline{X})\right]^2 .$$

Hence, (2.2.14) becomes

$$n^2 S_n(a) = \sum_{k=1}^{n} \left[N_k(\underline{X}) - P_k(\underline{X})\right]^2 - a\left[N_n(\underline{X}) - P_n(\underline{X})\right]^2$$

$$+ (a^2 - a) \sum_{k=1}^{n} \left[\delta_k(\underline{X})\right]^2 ,$$

which reduces to (2.2.13) in view of the fact that $\left[\delta_k(\underline{X})\right]^2 = 1$ with probability one, for k=1,2,...,n. This completes the proof of the theorem.

An intuitively reasonable criterion for choosing a representative statistic from each class may be based upon the preceding corollary. With respect to $S_n^{(a)}$, (1-a) may be regarded as the weight assigned to $\left[N_n(\underline{X}) - P_n(\underline{X})\right]^2$, whereas, for $R_n^{(a)}$, the corresponding weight assigned to $\left[N_n(\underline{X}^{-1}) - P_n(\underline{X}^{-1})\right]^2$ is (a). It is not clear how one would justify the assignment to $\left[N_n(\underline{X}) - P_n(\underline{X})\right]^2$ or $\left[N_n(\underline{X}^{-1}) - P_n(\underline{X}^{-1})\right]^2$ a weight different from that assigned to $\left[N_k(\underline{X}) - P_k(\underline{X})\right]^2$ or $\left[N_k(\underline{X}^{-1}) - P_k(\underline{X}^{-1})\right]^2$, k=1,2,...,n-1, (i.e., 1). Since $S_n^{(0)}$ and $R_n^{(1)}$ are the two statistics in which the full weight of 1 is assigned to the nth term, they appear to be logical choices in determining representative statistics from the two classes. In addition, exact critical values are available for both $S_n^{(0)}$ and $R_n^{(1)}$, and these two statistics are somewhat easier to compute than other members of their respective classes.

2.3 A Third Class of Cramér-von Mises Type Statistics

Both invariance properties defined in Chapter 1 are desirable properties for tests of symmetry. In Section 2.2, it was shown that $S_n^{(a)}$ and $R_n^{(a)}$ possess Invariance Property I for all $a \in [0,1]$. However, it is easily seen that neither class contains a statistic possessing Invariance Property II. By Corollary 2.2.1, the assumption that either $R_n^{(1-a)}$ or $S_n^{(a)}$ possesses this property leads to the contradiction that $S_n^{(a)}$ is equivalent to $R_n^{(1-a)}$. Corollary 2.2.1 does suggest, however, a method of combining statistics of the R_n class with those of the S_n class to produce statistics possessing both invariance properties, as shown by the following theorem.

Theorem 2.3.1:

Let

$$T_n^{(a)} \equiv T_n^{(a)}(\underline{X}) = \frac{1}{2}\left[R_n^{(1-a)}(\underline{X}) + S_n^{(a)}(\underline{X})\right].$$

Then, for every $a \in [0,1]$, $T_n^{(a)}$ possesses invariance properties I and II, and leads to a consistent test of symmetry against any alternative in $\Phi - \Phi_s$.

Proof:

That $T_n^{(a)}$ possesses Invariance Property I is clear, as both $R_n^{(1-a)}$ and $S_n^{(a)}$ possess the property. Invariance under the transformation $\underline{X} \to \underline{X}^{-1}$ follows from Corollary 2.2.1, since, with probability one,

$$\begin{aligned} T_n^{(a)}(\underline{X}) &= \frac{1}{2}\left[S_n^{(a)}(\underline{X}) + R_n^{(1-a)}(\underline{X})\right] \\ &= \frac{1}{2}\left[R_n^{(1-a)}(\underline{X}^{-1}) + S_n^{(a)}(\underline{X}^{-1})\right] \\ &= T_n^{(a)}(\underline{X}^{-1}). \end{aligned}$$

Finally, the consistency of the test based on $T_n^{(a)}$ follows immediately from (2.2.12) and the corresponding result for $S_n^{(a)}$.

For the reasons stated at the end of Section 2.2, $T_n^{(0)}$ appears to be a logical choice of the "best" statistic from the class $\left\{T_n^{(a)} : 0 \le a \le 1\right\}$. Table 1 provides the critical values for $T_n^* = \frac{n^2}{4} T_n^{(0)}$ at selected levels, α, and sample sizes, $n = 10(1)24$. Because the exact null distribution is not available in closed form for either $R_n^{(a)}$ or $S_n^{(a)}$, the exact distribution of $T_n^{(0)}$ was calculated by means of a computer enumeration of all possible values of the statistic, one for each $\underline{d}$ in Δ. Since Δ contains 2^n sample points, this method is suitable only

TABLE 1

EXACT CRITICAL VALUES FOR $T^* = \frac{n^2}{4} T_n^{(0)}$

n = 10		n = 11		n = 12	
x	$P(T^* \leq x)$	x	$P(T^* \leq x)$	x	$P(T^* \leq x)$
32.25	0.8945	33.00	0.8955	43.00	0.8970
33.25	0.9102	34.50	0.9072	44.00	0.9087
38.25	0.9492	49.50	0.9443	53.00	0.9497
39.25	0.9570	50.00	0.9502	54.00	0.9517
44.25	0.9746	56.50	0.9736	71.50	0.7961
48.25	0.9785	57.50	0.9775	73.50	0.7961
60.25	0.9863	81.50	0.9893	80.50	0.9883
62.25	0.9902	82.00	0.9912	83.50	0.9912

n = 13		n = 14		n = 15	
x	$P(T^* \leq x)$	x	$P(T^* \leq x)$	x	$P(T^* \leq x)$
49.25	0.8982	54.75	0.8979	68.00	0.8984
51.75	0.9009	55.75	0.9000	68.50	0.9001
64.75	0.9490	79.75	0.9497	84.00	0.9492
65.75	0.9504	80.75	0.9512	84.50	0.9501
77.75	0.9736	91.75	0.9736	112.50	0.9743
78.75	0.9751	92.75	0.9751	113.00	0.9750
101.75	0.9883	125.75	0.9899	132.50	0.9899
104.25	0.9902	127.72	0.9907	133.00	0.9902

n = 16		n = 17		n = 18	
x	$P(T^* \leq x)$	x	$P(T^* \leq x)$	x	$P(T^* \leq x)$
71.50	0.8988	83.25	0.8989	96.25	0.8989
72.50	0.9013	83.74	0.9009	97.25	0.9008
99.50	0.9487	115.25	0.9499	121.25	0.9493
100.00	0.9509	115.75	0.9502	122.25	0.9506
120.00	0.9748	137.75	0.9749	161.25	0.9746
121.00	0.9752	138.25	0.9753	162.25	0.9752
155.50	0.9897	184.75	0.9900	192.25	0.9898
156.50	0.9903	185.25	0.9903	193.25	0.9907

TABLE 1 (Continued)

n = 19		n = 20		n = 21	
x	$P(T^* \leq x)$	x	$P(T^* \leq x)$	x	$P(T^* \leq x)$
101.50	0.8990	117.00	0.8981	125.25	0.8998
102.00	0.9004	117.50	0.9003	126.25	0.9001
141.50	0.9495	152.00	0.9496	167.75	0.9496
142.00	0.9505	152.50	0.9503	168.25	0.9504
169.50	0.9747	192.50	0.9748	218.25	0.9749
170.00	0.9750	193.00	0.9753	218.75	0.9751
220.00	0.9897	253.00	0.9900	263.75	0.9899
220.50	0.9900	253.50	0.9902	264.25	0.9900

n = 22		n = 23		n = 24	
x	$P(T^* \leq x)$	x	$P(T^* \leq x)$	x	$P(T^* \leq x)$
137.73	0.8987	155.75	0.8998	163.25	0.8996
138.75	0.9006	156.25	0.9003	163.75	0.9007
189.75	0.9499	199.25	0.9497	222.75	0.9498
190.75	0.9506	199.75	0.9501	223.25	0.9505
227.75	0.9747	257.25	0.9748	280.75	0.9750
228.75	0.9751	257.75	0.9750	281.25	0.9751
296.75	0.9900	332.25	0.9900	348.25	0.9900
297.75	0.9901	332.75	0.9901	348.75	0.9901

for small sample sizes. Also, as noted above, the problem of deriving the asymptotic null distribution of $T_n^{(0)}$ does not appear to have an easy solution. However, there is some evidence indicating that the critical values corresponding to n = 24 will work fairly well for sample sizes larger than 24. This can be seen from Table 2, which gives the exact level, α'_m, resulting from the use of the correct α-level critical value (i.e., the critical value with exact level closest to α) corresponding to samples of size m, m = 15(1)24, when the sample size is actually 24. Also given is $\alpha'_m - \alpha_{24}$, where α_{24} is obtained by using the correct α-level critical value for samples of size 24. As can be seen from Table 2, this difference is fairly small.

TABLE 2

EXACT LEVELS OF TESTS FOR SAMPLES OF SIZE 24 BASED ON THE α-LEVEL CRITICAL VALUES OF $T_m^{(0)}$ $(15 \leq m \leq 24)$

	$\alpha = .05$		$\alpha = .01$	
m	α'_m	$\alpha'_m - \alpha_{24}$	α'_m	$\alpha'_m - \alpha_{24}$
15	0.0549	+0.0047	0.0112	+0.0012
16	0.0457	-0.0045	0.0097	-0.0003
17	0.0443	-0.0059	0.0080	-0.0020
18	0.0542	+0.0040	0.0109	+0.0009
19	0.0472	-0.0030	0.0095	-0.0005
20	0.0518	+0.0016	0.0082	-0.0018
21	0.0518	+0.0016	0.0102	+0.0002
22	0.0472	-0.0030	0.0093	-0.0007
23	0.0542	+0.0040	0.0084	-0.0016
24	0.0502	0.0	0.0100	0.0

CHAPTER 3

A TEST OF SYMMETRY BASED ON THE WATSON STATISTIC

3.1 A Class of Symmetry Statistics Based on the Watson Statistic

Although $T_n^{(0)}$, the statistic described in Section 2.3, is desirable from an invariance point of view, it does not appear that its properties will be easy to determine analytically. As noted in the previous chapter, its asymptotic null distribution is not available at this time. For this reason, an alternative statistic for testing H_o will be developed and studied in this chapter. This statistic will be based on μ_n^2, the two-sample statistic proposed by Watson [20]. A definition of μ_n^2 appears in Section 1.1.

The statistic μ_n^2 and its goodness-of-fit version μ_n^{gf} [19], possess several desirable properties. Both statistics are particularly useful when dealing with populations having circular distributions. Stephens [12] compared the goodness-of-fit test based on μ_n^{gf} with four other EDF-based goodness-of-fit tests using a Monte Carlo study. This study showed that the Watson statistic performed best against shifts in variance. In this chapter several nice properties of the modified version of μ_n^2 for testing symmetry will be established.

The procedure used to modify the Cramér-von Mises two-sample statistic, ω_n^2, can also be used to modify μ_n^2. Thus, the expression

$\left[F_n(y) - G_n(y)\right]$ in μ_n^2 may be replaced by either $\left[F_n^{(a)}(y) + F_n^{(a)}(-y) - 1\right]$ or $\left[F_n^{(a)}(y) + F_n^{(a)}(-y) - 2F_n(0)\right]$ to produce two classes, one corresponding to each of the above substitutions. However, the two resulting classes are equivalent. To see this, first note that for any constant C,

$$\int_{-\infty}^{+\infty} C \, dF_n(y) = C.$$

Thus, substituting in μ_n^2 any expression of the form

$$F_n^{(a)}(y) + F_n^{(a)}(-y) - C,$$

where C is constant with respect to the variable of integration, will result in a statistic independent of the value of C. For the sake of convenience in proving several of the results which follow, members of this class of symmetry statistics will be defined with C = 1.

Definition 3.1.1:

For any $a \in [0,1]$, the statistic $U_n^{(a)}$ is defined by

$$U_n^{(a)} = n \int_{-\infty}^{+\infty} \left[F_n^{(a)}(y) + F_n^{(a)}(-y) - 1 - \int_{-\infty}^{+\infty} (F_n^{(a)}(w) + F_n^{(a)}(-w) - 1) \, dF_n(w)\right]^2 dF_n(y)$$

where $F_n^{(a)}$ is given by Definition 2.1.1.

Note that a reasonable test of symmetry is obtained by rejecting H_o for "large" values of $U_n^{(a)}$.

Corresponding to (2.2.3) and (2.2.4), the following theorem gives two expressions for $U_n^{(a)}$ in terms of the functions $N_k(\cdot)$, $P_k(\cdot)$

and $\delta_k(\cdot)$. These will provide a convenient form for computation of $U_n^{(a)}$, and will be useful in establishing several properties of $U_n^{(a)}$.

Theorem 3.1.1:

Let the functions $\delta_k(\cdot)$, $P_k(\cdot)$ and $N_k(\cdot)$ be given by (1.1.5), (1.1.6) and (1.1.7), respectively. Then, with probability one,

$$U_n^{(a)}(\underline{X}) = n^{-2}\Bigg\{\sum_{k=1}^{n}\Big[N_k(\underline{X}) - P_k(\underline{X}) + a\delta_k(\underline{X})\Big]^2 - n^{-1}\Big[\sum_{k=1}^{n}(N_k(\underline{X}) - P_k(\underline{X}) + a\delta_k(\underline{X}))\Big]^2\Bigg\}$$

$$= n^{-2}\Bigg\{\sum_{k=1}^{n}\Big[N_k(\underline{X}^{-1}) - P_k(\underline{X}^{-1}) + (1-a)\delta_k(\underline{X}^{-1})\Big]^2 - n^{-1}\Big[\sum_{k=1}^{n}(N_k(\underline{X}^{-1}) - P_k(\underline{X}^{-1}) + (1-a)\delta_k(\underline{X}^{-1}))\Big]^2\Bigg\}. \quad (3.1.1)$$

Proof:

Clearly, $U_n^{(a)}$ can be expressed as

$$U_n^{(a)} = \sum_{i=1}^{n}\Big[F_n^{(a)}(X_i) + F_n^{(a)}(-X_i) - 1\Big]^2 - n^{-1}\Big[\sum_{j=1}^{n}(F_n^{(a)}(X_j) + F_n^{(a)}(-X_j) - 1)\Big]^2$$

$$= \sum_{i=1}^{n}\Big[F_n^{(a)}(X_i) + F_n^{(a)}(-X_i) - 2F_n(0)\Big]^2 - n^{-1}\Big[\sum_{j=1}^{n}(F_n^{(a)}(X_j) + F_n^{(a)}(-X_j) - 2F_n(0))\Big]^2.$$

The theorem then follows from the two equalities, (2.2.5) and (2.2.9), established in proving Theorem 2.2.2.

3.2 Properties of $U_n^{(a)}$

Due to the close relationship between μ_n^2 and ω_n^2, it is not surprising that the properties of $U_n^{(a)}$ are similar to those of $R_n^{(a)}$ and $S_n^{(a)}$ (see Chapter 2). For example, as a consequence of Theorem 3.1.1, $U_n^{(a)}$ depends on $\underline{X}$ through $\underline{\delta}(\underline{X})$ alone, and, hence, $U_n^{(a)}$ is distribution free (see remarks after Lemma 2.2.1). The next theorem, analogous to Theorem 2.2.1, establishes Invariance Property I for $U_n^{(a)}$.

Theorem 3.2.1:

For any $a \in [0,1]$,

$$U_n^{(a)}(\underline{X}) = U_n^{(a)}(-\underline{X}) \quad \text{w.p. 1.} \tag{3.2.1}$$

Proof:

From Definition 1.1.2, it is easily seen that, with probability one,

$$N_k(\underline{X}) = P_k(-\underline{X}),$$

and

$$\delta_k(\underline{X}) = -\delta_k(-\underline{X}).$$

Hence, (3.2.1) follows directly from (3.1.1), and the proof is complete.

The following theorem is a direct consequence of Theorem 3.1.1 and is stated without proof.

Theorem 3.2.2:

For every $a \in [0,1]$,

$$U_n^{(a)}(\underline{X}) = U_n^{(1-a)}(\underline{X}^{-1}) \quad \text{w.p. 1,}$$

and, in particular,

$$U_n^{(\frac{1}{2})}(\underline{X}) = U_n^{(\frac{1}{2})}(\underline{X}^{-1}) \quad \text{w.p. 1.}$$

Thus, $U_n^{(\frac{1}{2})}$ possesses both invariance properties I and II, and, therefore, is the logical choice as a "best" statistic from among those in the class $\left\{U_n^{(a)} : 0 \le a \le 1\right\}$. The next theorem shows that all statistics in this class are asymptotically equivalent in the sense that the difference between any two statistics converges to zero with probability one.

Theorem 3.2.3:

Under H_o, $U_n^{(a)}$ and $U_n^{(b)}$ have the same asymptotic distribution for any $a,b \in [0,1]$. In fact,

$$|U_n^{(a)} - U_n^{(b)}| \to 0 \quad \text{w.p. 1} \quad \text{as } n \to \infty.$$

Proof:

For any $a \in [0,1]$, let

$$A_n^{(a)}(y) = F_n^{(a)}(y) + F_n^{(a)}(-y) - 1$$

$$- \int_{-\infty}^{\infty} \left[F_n^{(a)}(w) + F_n^{(a)}(-w) - 1\right] dF_n(w).$$

Then, for any $b \in [0,1]$,

$$U_n^{(a)} = n\int_{-\infty}^{+\infty}\left[A_n^{(a)}(y)\right]^2 dF_n(y)$$

$$= n\int_{-\infty}^{+\infty}\left[(A_n^{(a)}(y) - A_n^{(b)}(y)) + A_n^{(b)}(y)\right]^2 dF_n(y)$$

$$= n\int_{-\infty}^{+\infty}\left[A_n^{(a)}(y) - A_n^{(b)}(y)\right]^2 dF_n(y) + U_n^{(b)}$$

$$+ 2n\int_{-\infty}^{+\infty}\left[A_n^{(a)}(y) - A_n^{(b)}(y)\right]\left[A_n^{(b)}(y)\right] dF_n(y).$$

Hence,

$$|U_n^{(a)} - U_n^{(b)}| \le n\int_{-\infty}^{+\infty}\left[\sup_y |A_n^{(a)}(y) - A_n^{(b)}(y)|^2\right] dF_n(y)$$

$$+ 2n\int_{-\infty}^{+\infty} \sup_y |A_n^{(a)}(y) - A_n^{(b)}(y)| \cdot \sup_y |A_n^{(b)}(y)|\, dF_n(y)$$

$$= n\left\{\sup_y |A_n^{(a)}(y) - A_n^{(b)}(y)|\right\}^2$$

$$+ 2n \sup_y |A_n^{(a)}(y) - A_n^{(b)}(y)| \cdot \sup_y |A_n^{(b)}(y)|. \quad (3.2.2)$$

Now,

$$\sup_y |A_n^{(a)}(y) - A_n^{(b)}(y)| \le 2\left[\sup_y |F_n^{(a)}(y) - F_n^{(b)}(y)|\right.$$

$$\left. + \sup_y |F_n^{(a)}(-y) - F_n^{(b)}(-y)|\right].$$

Since

$$\sup_y |F_n^{(a)}(y) - F_n^{(b)}(y)| \le n^{-1},$$

it follows that

$$\sup_y |A_n^{(a)}(y) - A_n^{(b)}(y)| \le 4n^{-1}.$$

Therefore, from (3.2.2),

$$|U_n^{(a)} - U_n^{(b)}| \leq n(\tfrac{4}{n})^2 + 2n(\tfrac{4}{n}) \sup_y |A_n^{(b)}(y)|$$

$$= o_n(1) + 8 \sup_y |A_n^{(b)}(y)|.$$

Thus the proof will be complete if it can be shown that $\sup_y |A_n^{(b)}(y)|$ converges to zero with probability one as n tends to infinity. But this follows from Corollary 2.2.1, since

$$\sup_y |A_n^{(b)}(y)| \leq 2 \sup_y |F_n^{(b)}(y) + F_n^{(b)}(-y) - 1|.$$

Note that $U_n^{(\frac{1}{2})}$ may also be regarded as a member of the class $\left\{\frac{1}{2}\left[U_n^{(a)} + U_n^{(1-a)}\right] : 0 \leq a \leq 1\right\}$, and that all statistics in this class possess both invariance properties. However, since

$$|\tfrac{1}{2}\left[U_n^{(a)} + U_n^{(1-a)}\right] - U_n^{(b)}| \leq \tfrac{1}{2}|U_n^{(a)} - U_n^{(b)}| + \tfrac{1}{2}|U_n^{(1-a)} - U_n^{(b)}|,$$

Theorem 3.2.3 implies that, under H_o, statistics in this class are asymptotically equivalent to any statistic in the U_n class. For this reason, the class $\left\{\frac{1}{2}\left[U_n^{(a)} + U_n^{(1-a)}\right] : 0 \leq a \leq 1\right\}$ is not given further consideration.

The final theorem presented in this section establishes the consistency of tests of symmetry based on $U_n^{(a)}$.

Theorem 3.2.4:

For every $a \in [0,1]$, $U_n^{(a)}$ is consistent against any alternative in $\Phi - \Phi_s$.

Proof:

Observe that, from (1.1.1) and the continuity of F,

$$\Gamma = \int_{-\infty}^{+\infty} \left[F(y) + F(-y) - 1 - \int_{-\infty}^{+\infty} (F(w) + F(-w) - 1)\ dF(w)\right]^2 dF(y)$$

is positive if and only if $F \in \Phi - \Phi_s$. If it can be shown that for any $a \in [0,1]$,

$$\lim_{n \to \infty} n^{-1} U_n^{(a)} = \Gamma \quad \text{w.p. 1,} \tag{3.2.3}$$

it will then follow that with probability one, $U_n^{(a)}$ tends to infinity with n if and only if $F \in \Phi - \Phi_s$, which will imply the consistency of tests based on $U_n^{(a)}$. Thus it is only necessary to establish (3.2.3). Let $H_n(y) = F_n^{(a)}(y) - F(y)$. Then,

$$n^{-1} U_n^{(a)} = \int_{-\infty}^{+\infty} \left[H_n(y) + H_n(-y) + (F(y) + F(-y) - 1)\right]^2 dF_n(y)$$

$$- \left\{\int_{-\infty}^{+\infty} \left[H_n(w) + H_n(-w) + (F(w) + F(-w) - 1)\right]\ dF_n(w)\right\}^2. \tag{3.2.4}$$

Upon expanding the integrand and rearranging terms, the first integral in (3.2.4) becomes

$$\int_{-\infty}^{+\infty} \left[H_n^2(y) + H_n^2(-y) + 2(H_n(y) + H_n(-y))(F(y) + F(-y) - 1)\right.$$

$$\left. + 2\, H_n(y)\, H_n(-y)\right]\ dF_n(y) + \int_{-\infty}^{+\infty} \left[F(y) + F(-y) - 1\right]^2 dF_n(y). \tag{3.2.5}$$

The two integrals in (3.2.5) will now be considered individually. The absolute value of the first integral is bounded above by

$$\left[\sup_y |H_n(y)|\right]^2 + \left[\sup_y |H_n(-y)|\right]^2 + 2\left[\sup_y |H_n(y)| + \sup_y |H_n(-y)|\right]$$

$$\times \sup_y |F(y) + F(-y) - 1| + 2 \sup_y |H_n(y)| \cdot \sup_y |H_n(-y)|. \qquad (3.2.6)$$

From Lemma 2.1.1 and the fact that

$$\sup_y |F(y) + F(-y) - 1| \le 1,$$

it follows that (3.2.6) and hence, the first integral in (3.2.5) converge to zero with probability one as n tends to infinity. The second integral in (3.2.5) can be expressed as

$$n^{-1} \sum_{i=1}^{n} \left[F(X_i) + F(-X_i) - 1\right]^2. \qquad (3.2.7)$$

Since

$$\int_{-\infty}^{+\infty} |F(y) + F(-y) - 1|^2 \, dF(y) \le 1,$$

the strong law of large numbers implies that (3.2.7) converges with probability one to

$$E\left[(F(X) + F(-X) - 1)^2\right] = \int_{-\infty}^{+\infty} \left[F(y) + F(-y) - 1\right]^2 dF(y). \qquad (3.2.8)$$

Thus the expression given by (3.2.5), as n tends to $+\infty$, converges with probability one to the right-hand side of (3.2.8). A similar argument shows that

$$\int_{-\infty}^{+\infty} \left[H_n(w) + H_n(-w) + (F(w) + F(-w) - 1)\right] dF_n(w)$$

converges with probability one to

$$\int_{-\infty}^{+\infty} \left[F(w) + F(-w) - 1\right] dF(w).$$

Hence, the second integral on the right-hand side of (3.2.4) converges with probability one to

$$\left\{\int_{-\infty}^{+\infty}\left[F(w) + F(-w) - 1\right]\ dF(w)\right\}^2.$$

This establishes (3.2.3) and completes the proof of the theorem.

3.3 The Asymptotic Null Distribution of $U_n^{(a)}$

In this section, the common asymptotic null distribution of the statistics in the class $\left\{U_n^{(a)} : 0 \le a \le 1\right\}$ will be derived by considering a related random variable, $\tilde{U}_n$, defined as

$$\tilde{U}_n = n \int_{-\infty}^{+\infty}\left[F_n(y) + F_n(-y) - 1 - \int_{-\infty}^{+\infty}(F_n(w) + F_n(-w) - 1)\ dF(w)\right]^2 dF(y).$$

Note that, because of the continuity of F, $\tilde{U}_n$ is independent of the version of the EDF (F_n or $F_n^{(a)}$, $0 \le a \le 1$) appearing in the integrand. Theorem 3.3.1 below shows that $\tilde{U}_n$ and $U_n^{(\frac{1}{2})}$ have the same asymptotic null distribution. The proof is based on the corresponding proof given by Watson [20] for μ_n^2. Before stating this theorem, two lemmas necessary for its proof will be established.

Lemma 3.3.1:

Let F_n denote the EDF based on a sample of size n from $F \in \Phi_s$, and let

$$V_n = n^{\frac{1}{2}} \int_{-\infty}^{+\infty}\left[F_n(y) + F_n(-y) - 1\right]\ dF(y). \qquad (3.3.1)$$

Then, as n tends to infinity, V_n converges in law to a normal random variable with mean zero and variance 1/3.

Proof:

Clearly

$$V_n = n^{-\frac{1}{2}} \int_{-\infty}^{+\infty} \left\{ \sum_{i=1}^{n} I\left[X_i \le y\right] + \sum_{i=1}^{n} I\left[X_i \le -y\right] - n \right\} dF(y)$$

$$= n^{-\frac{1}{2}} \sum_{i=1}^{n} \left\{ \int_{X_i}^{+\infty} dF(y) + \int_{-\infty}^{-X_i} dF(y) \right\} - n^{\frac{1}{2}}$$

$$= n^{-\frac{1}{2}} \sum_{i=1}^{n} \left\{ \int_{F(X_i)}^{1} dt + \int_{0}^{F(-X_i)} dt \right\} - n^{\frac{1}{2}}$$

$$= n^{-\frac{1}{2}} \sum_{i=1}^{n} \left[1 - F(X_i) + F(-X_i) \right] - n^{\frac{1}{2}} .$$

Since $F(-X_i) = 1 - F(X_i)$ if $F \in \Phi_s$, the last expression above equals

$$n^{\frac{1}{2}} \left[1 - 2n^{-1} \sum_{i=1}^{n} F(X_i) \right] . \qquad (3.3.2)$$

Now, since $\{F(X_1), F(X_2), \ldots, F(X_n)\}$ is distributed as a random sample from a uniform distribution on [0,1], the lemma follows from (3.3.2) and the Central Limit Theorem. This proof is therefore complete.

The second lemma needed for the proof of Theorem 3.3.1 is now established.

Lemma 3.3.2:

Let

$$C_n = V_n - n^{\frac{1}{2}} \int_{-\infty}^{+\infty} \left[F_n^{(\frac{1}{2})}(y) + F_n^{(\frac{1}{2})}(-y) - 1 \right] dF_n(y), \qquad (3.3.3)$$

where V_n is given by (3.3.1). Then, for every $F \in \Phi_s$,

$$E(C_n) = 0, \tag{3.3.4}$$

and

$$\mathrm{Var}(C_n) \to 0 \text{ as } n \to \infty. \tag{3.3.5}$$

Proof:

Note that the second term on the right-hand side of (3.3.3) is equal to V_n with $dF(y)$ replaced by $dF_n(y)$. Using (3.3.2), with probability one,

$$\begin{aligned} C_n &= n^{\frac{1}{2}}\Big[1 - 2n^{-1}\sum_{i=1}^{n} F(X_i)\Big] \\ &\quad - n^{\frac{1}{2}}\left\{n^{-1}\sum_{i=1}^{n}\left(F_n^{(\frac{1}{2})}(X_i) + n^{-1}\sum_{j=1}^{n} I\Big[X_j \le -X_i\Big]\right)\right\} + n^{\frac{1}{2}} \\ &= n^{\frac{1}{2}}\left\{2 - 2n^{-1}\sum_{i=1}^{n} F(X_i) - n^{-2}\sum_{i=1}^{n}\left(nF_n^{(\frac{1}{2})}(X_i)\right.\right. \\ &\quad \left.\left. + \sum_{j=1}^{n} I\Big[X_i + X_j \le 0\Big]\right)\right\}. \end{aligned}$$

Since, with probability one,

$$n\sum_{i=1}^{n} F_n^{(\frac{1}{2})}(X_i) = \sum_{i=1}^{n}(i - \tfrac{1}{2}) = \tfrac{1}{2}n^2,$$

it follows that

$$C_n = n^{\frac{1}{2}}\left\{\frac{3}{2} - 2n^{-1}\sum_{i=1}^{n} F(X_i) - n^{-2}\sum_{i=1}^{n}\sum_{j=1}^{n} I\Big[X_i + X_j \le 0\Big]\right\}. \tag{3.3.6}$$

Therefore,

$$E(C_n) = n^{\frac{1}{2}}\left\{\frac{3}{2} - 2n^{-1}\sum_{i=1}^{n} E(F(X_i)) - n^{-2}\sum_{i=1}^{n}\sum_{j=1}^{n} E\Big(I\Big[X_i + X_j \le 0\Big]\Big)\right\}. \tag{3.3.7}$$

Since $F(X_i)$ has a uniform distribution on the unit interval,

$$E(F(X_i)) = \frac{1}{2}, \quad i=1,2,\ldots,n. \tag{3.3.8}$$

Using (2.2.10), it can be easily shown that if $F \in \Phi_s$, $(X_i + X_j)$ is distributed symmetrically about zero. Thus, for $i,j=1,2,\ldots,n$,

$$E(I[X_i + X_j \le 0]) = P(X_i + X_j \le 0) = \frac{1}{2}. \tag{3.3.9}$$

Substituting (3.3.8) and (3.3.9) in (3.3.7) yields

$$E(C_n) = n^{\frac{1}{2}} \left\{ \frac{3}{2} - 2n^{-1}(\tfrac{1}{2}n) - n^{-2}(\tfrac{1}{2}n^2) \right\} = 0,$$

thus establishing (3.3.4).

Now, consider the variance of C_n. Squaring the expression given by (3.3.6) gives

$$\begin{aligned} C_n^2 = n \Bigg\{ & \frac{9}{4} + 4n^{-2}\left[\sum_{i=1}^{n} F(X_i)\right]^2 + n^{-4}\left(\sum_{i=1}^{n}\sum_{j=1}^{n} I\left[X_i + X_j \le 0\right]\right)^2 \\ & - 6n^{-1}\sum_{i=1}^{n} F(X_i) - 3n^{-2}\sum_{i=1}^{n}\sum_{j=1}^{n} I\left[X_i + X_j \le 0\right] \\ & + 4n^{-3}\sum_{i=1}^{n} F(X_i)\sum_{j=1}^{n}\sum_{k=1}^{n} I\left[X_j + X_k \le 0\right] \Bigg\}. \end{aligned}$$

Taking expectations on both sides and using (3.3.8) and (3.3.9) results in the following expression for the variance of C_n:

$$\begin{aligned} \mathrm{Var}(C_n) = {} & \frac{-9}{4}n + 4n^{-1}E\left\{\left[\sum_{i=1}^{n} F(X_i)\right]^2\right\} \\ & + n^{-3}E\left\{\left(\sum_{i=1}^{n}\sum_{j=1}^{n} I\left[X_i + X_j \le 0\right]\right)^2\right\} \\ & + 4n^{-2}E\left\{\sum_{i=1}^{n} F(X_i)\sum_{j=1}^{n}\sum_{k=1}^{n} I\left[X_j + X_k \le 0\right]\right\}. \end{aligned} \tag{3.3.10}$$

The three expectations in (3.3.10) are now considered individually.

As noted earlier, $\{F(X_1),F(X_2),\ldots,F(X_n)\}$ is distributed as a random sample from a uniform distribution on the unit interval. Therefore, the first expectation in (3.3.10) is given by

$$E\left\{\left[n^{-1}\sum_{i=1}^{n}F(X_i)\right]^2\right\} = \mathrm{Var}\left(n^{-1}\sum_{i=1}^{n}F(X_i)\right) + \left\{E\left(n^{-1}\sum_{i=1}^{n}F(X_i)\right)\right\}^2$$

$$= 12n^{-1} + \frac{1}{4}. \qquad (3.3.11)$$

To evaluate the second expectation in (3.3.10), let α_{ij} denote $I[X_i + X_j \le 0]$. Then

$$\left\{\sum_{i=1}^{n}\sum_{j=1}^{n} I\left[X_i+X_j \le 0\right]\right\}^2 = \sum_{i=1}^{n}\alpha_{ii}^2 + \sum_{i=1}^{n}\sum_{j=1}^{n}(\alpha_{ij}^2 + \alpha_{ij}\alpha_{ji}) + \sum_{i=1}^{n}\sum_{j=1}^{n}\alpha_{ii}\alpha_{jj}$$

$$+ \sum_{\substack{i=1\\ i\neq j}}^{n}\sum_{j=1}^{n}(\alpha_{ii}\alpha_{ij} + \alpha_{ii}\alpha_{ji} + \alpha_{ij}\,\alpha_{ii} + \alpha_{ji}\alpha_{ii})$$

$$+ \sum_{i=1}^{n}\sum_{\substack{j=1\\ i\neq j\neq k}}^{n}\sum_{k=1}^{n}(\alpha_{ii}\alpha_{jk} + \alpha_{jk}\alpha_{ii})$$

$$+ \sum_{i=1}^{n}\sum_{\substack{j=1\\ i\neq j\neq k}}^{n}\sum_{k=1}^{n}(\alpha_{ij}\alpha_{ik} + \alpha_{ij}\alpha_{ki} + \alpha_{ji}\alpha_{ik} + \alpha_{ji}\alpha_{ki})$$

$$+ \sum_{i=1}^{n}\sum_{\substack{j=1\\ i\neq j\neq k\neq \ell}}^{n}\sum_{k=1}^{n}\sum_{\ell=1}^{n}\alpha_{ij}\alpha_{k\ell}.$$

Since $\alpha_{ij} = \alpha_{ji}$ for every pair of integers, i and j, and since the terms within a particular sum all have identical expectations, the right-hand side of the above equality becomes, for $n \ge 4$,

$$nE(\alpha_{ii}^2) + 2n(n-1)E(\alpha_{ij}^2) + n(n-1)E(\alpha_{ii}\alpha_{jj})$$
$$+ 4n(n-1)E(\alpha_{ii}\alpha_{ij}) + 2n(n-1)(n-2)E(\alpha_{ii}\alpha_{jk})$$
$$+ 4n(n-1)(n-2)E(\alpha_{ij}\alpha_{ik}) + n(n-1)(n-2)(n-3)E(\alpha_{ij}\alpha_{k\ell}), \quad (3.3.12)$$

where i, j, k and ℓ denote four distinct positive integers less than or equal to n. Now, the expectation in (3.3.12) may be evaluated as follows. First, since $\alpha_{ij}^2 = \alpha_{ij}$ and $\alpha_{ii}^2 = \alpha_{ii}$, (3.3.9) implies that

$$E(\alpha_{ij}^2) = E(\alpha_{ij}) = E(\alpha_{ii}^2) = E(\alpha_{ii}) = \frac{1}{2}. \quad (3.3.13)$$

Next, since α_{ij} is independent of $\alpha_{k\ell}$ and α_{ii} is independent of α_{jj} and α_{jk}, it follows that

$$E(\alpha_{ii}\alpha_{jj}) = E(\alpha_{ij}\alpha_{k\ell}) = E(\alpha_{ii}\alpha_{jk}) = \frac{1}{4}. \quad (3.3.14)$$

Finally,

$$E(\alpha_{ij}\alpha_{ik}) = P(X_i+X_j \le 0 \text{ and } X_i+X_k \le 0)$$

$$= \iiint_{\{(y_i+y_j\le 0)\cap(y_i+y_k\le 0)\}} dF(y_k)\, dF(y_j)\, dF(y_i).$$

$$\int_0^1 \int_0^{1-s} \int_0^{1-s} dt\, dw\, ds = \frac{1}{3} \quad (3.3.15)$$

and, similarly,

$$E(\alpha_{ii}\alpha_{ij}) = \frac{3}{8}. \quad (3.3.16)$$

Substituting (3.3.13), (3.3.14), (3.3.15) and (3.3.16) in (3.3.12),

$$E\left\{\left(\sum_{i=1}^{n}\sum_{j=1}^{n} I\left[X_i + X_j \le 0\right]\right)^2\right\} = n(\tfrac{1}{2}) + 2n(n-1)(\tfrac{1}{2}) + n(n-1)(\tfrac{1}{4})$$

$$+ 4n(n-1)(\tfrac{3}{8}) + 2n(n-1)(n-2)(\tfrac{1}{4})$$

$$+ 4n(n-1)(n-2)(\tfrac{1}{3}) + n(n-1)(n-2)(n-3)(\tfrac{1}{4})$$

$$= (\tfrac{1}{4})n^4 + (\tfrac{1}{3})n^3 + o(n^3). \qquad (3.3.17)$$

In a similar manner, the third expectation in (3.3.10) can be shown to be equal to

$$E\left(\sum_{i=1}^{n} F(X_i) \sum_{j=1}^{n}\sum_{k=1}^{n} I\left[X_j + X_k \le 0\right]\right) = (\tfrac{1}{4})n^3 - (\tfrac{1}{6})n^2 + o(n^2). \qquad (3.3.18)$$

Substituting (3.3.11), (3.3.17) and (3.3.18) in (3.3.10) yields

$$\mathrm{Var}(C_n) = (-\tfrac{9}{4})n + 4n\left\{(\tfrac{1}{12})n^{-1} + \tfrac{1}{4}\right\} + n^{-3}\left\{(\tfrac{1}{4})n^4 + (\tfrac{1}{3})n^3 + o(n^3)\right\}$$

$$+ 4n^{-2}\left\{(\tfrac{1}{4})n^3 - (\tfrac{1}{6})n^2 + o(n^2)\right\}$$

$$= n^{-3}o(n^3) + n^{-2}o(n^2),$$

which establishes (3.3.5) and completes the proof of the lemma.

Theorem 3.3.1 may now be given.

Theorem 3.3.1:

For any $F \in \Phi_s$, $\left|U_n^{(\frac{1}{2})} - \tilde{U}_n\right|$ converges to zero in probability as n tends to infinity.

Proof:

Expanding the integrands and rearranging terms yields

$$\left(U_n^{(\frac{1}{2})} - \tilde{U}_n\right) = n\left\{\int_{-\infty}^{+\infty}\left[F_n^{(\frac{1}{2})}(y) + F_n^{(\frac{1}{2})}(-y) - 1\right]^2 dF_n(y)\right.$$
$$\left. - \int_{-\infty}^{+\infty}\left[F_n(y) + F_n(-y) - 1\right]^2 dF(y)\right\}$$
$$+ n\left\{\left(\int_{-\infty}^{+\infty}\left[F_n(y) + F_n(-y) - 1\right] dF(y)\right)^2\right.$$
$$\left. - \left(\int_{-\infty}^{+\infty}\left[F_n^{(\frac{1}{2})}(y) + F_n^{(\frac{1}{2})}(-y) - 1\right] dF_n(y)\right)^2\right\}. \qquad (3.3.19)$$

The fact that the first term on the right-hand side of (3.3.19) converges to zero in probability has been established by Rothman and Woodroofe [15]. Using (3.3.1) and (3.3.3), the second term can be written as

$$V_n^2 - (V_n - C_n)^2 = C_n(2V_n - C_n).$$

Hence, by Lemmas 3.3.1 and 3.3.2, the second term on the right-hand side of (3.3.19) converges to zero in probability as n tends to infinity, and the proof is complete.

The asymptotic null distribution of $\tilde{U}_n$ may now be derived. It will be necessary at this time to consider some preliminary results. Proofs of Theorems 3.3.2 and 3.3.3, and Lemma 3.3.3 may be found in the reference indicated, and these results will be stated without proof. Proofs of the remainder of the preliminary results, Lemmas 3.3.4, 3.3.5, 3.3.6 and 3.3.7, will be given.

Theorem 3.3.2 (Prabhu [13], p. 28):

Let y(t) be a stochastic process on [0,1], with

$$E(|y(t)|^2) < +\infty$$

for every t in [0,1], and let K(s,t) denote the covariance kernel of the process, y(t). Then y(t) is Riemann integrable if and only if the integral

$$\int_0^1 \int_0^1 K(s,t)\, ds\, dt$$

exists. If the above integral does exist, then

$$E\left(\int_0^1 y(t)\, dt\right) = \int_0^1 E(y(t))\, dt,$$

and

$$E\left(\int_0^1 \int_0^1 y(s)\, y(t)\, ds\, dt\right) = \int_0^1 \int_0^1 K(s,t)\, ds\, dt.$$

The following definitions will be needed in several of the following results. Let D denote the metric space of functions on [0,1] that are right-continuous and have left-hand limits, and let Λ denote the class of strictly increasing, continuous mappings of [0,1] onto itself. The Skorokhod metric, d, may be defined as follows ([2], p. 111): for $x,y \in D$,

$$d(x,y) = \inf\left\{\varepsilon > 0 : \exists\ \lambda \in \Lambda \text{ such that } \sup_t |t - \lambda(t)| \le \varepsilon \text{ and } \sup_t |x(t) - y(\lambda(t))| \le \varepsilon\right\}.$$

Theorem 3.3.3 may now be stated.

Theorem 3.3.3 (Billingsley [2], p. 30):

If the sequence of stochastic processes $\{y_n(t);\ n=1,2,\ldots\}$ in D converges in law to the process y(t), and if $g(\cdot)$ is any measurable functional on D which is continuous in metric d almost everywhere (with

respect to the probability measure associated with y(t)), then as n tends to infinity, $g(y_n(t))$ converges in law to $g(y(t))$.

Lemma 3.3.3 (Durbin [7], p. 31):

The functional on D defined by

$$z(x(\cdot)) = \int_0^1 x(t)\, dt$$

is continuous in the Skorokhod metric, d.

Lemma 3.3.4:

Let Z(t) be a Riemann integrable Gaussian process on [0,1]. Then $\int_0^1 Z(t)dt$ is a normal random variable.

Proof:

Let

$$m(t) = E(Z(t)),$$

$$K(s,t) = Cov(Z(s),Z(t)),$$

and

$$W_n = \sum_{j=1}^{n} \left[t_j(n) - t_{j-1}(n)\right] Z(t_j(n)),$$

where $\{0=t_0(n) \le t_1(n) \le \cdots \le t_n(n) = 1;\ n=1,2,\ldots\}$ denotes any sequence of partitions of [0,1] such that $\max_j |t_j(n)-t_{j-1}(n)|$ converges to zero as n tends to infinity. Clearly, for each n, W_n is a normal random variable with

$$E(W_n) = \sum_{j=1}^{n} \left[t_j(n)-t_{j-1}(n)\right] m(t_j(n)),$$

and

$$Var(W_n) = \sum_{k=1}^{n} \sum_{j=1}^{n} \left[t_k(n)-t_{k-1}(n)\right] \left[t_j(n)-t_{j-1}(n)\right] K(t_k(n),t_j(n)).$$

By the definition of the Riemann integral, as n tends to infinity,

$$E(W_n) \to M = \int_0^1 m(t)\, dt,$$

and

$$Var(W_n) \to \sigma^2 = \int_0^1 \int_0^1 K(s,t)\, ds\, dt.$$

This implies that the characteristic function of W_n converges to the characteristic function of a normal random variable with mean, M, and variance, σ^2, and hence, that W_n converges in law to such a normal random variable. Since $\int_0^1 Z(t)\, dt$ is the limit (in mean square) of W_n as n tends to infinity, the proof is complete.

Lemma 3.3.5:

Let G_n denote the EDF corresponding to a random sample of size n from a uniform distribution on [0,1]. If the processes $Y_n^*(t)$ and $Q_n^*(t)$ are defined by

$$Y_n^*(t) = n^{\frac{1}{2}}\left[G_n\left(\frac{1+t}{2}\right) - \left(\frac{1+t}{2}\right)\right], \quad -1 \le t \le 1,$$

and

$$Q_n^*(t) = Y_n^*(t) + Y_n^*(-t) - \int_0^1 \left[Y_n^*(w) + Y_n^*(-w)\right] dw, \quad 0 \le t \le 1,$$

then $Q_n^*(t)$ converges in law to a Gaussian process on [0,1].

Proof:

It is a well-known result (see [2], p. 141) that the process $Y_n(t)$ defined by

$$Y_n(t) = n^{\frac{1}{2}}(G_n(t) - t)$$

converges in law to a Brownian bridge process, Y(t), on [0,1]. Clearly,

Y_n^* is the process obtained from Y_n be rescaling to the interval [-1,1]. Hence Y_n^* converges in law to Y*, where Y* is the process on [-1,1] defined by

$$Y^*(t) = Y\left(\frac{1+t}{2}\right).$$

Since Y is a Brownian bridge process, Y* is a Gaussian process on [-1,1] with

$$E(Y^*(t)) = E(Y\left(\frac{1+t}{2}\right)) = 0, \quad -1 \le t \le 1 \tag{3.3.20}$$

and

$$K^*(s,t) = E(Y^*(s)Y^*(t)) = E(Y\left(\frac{1+s}{2}\right)Y\left(\frac{1+t}{2}\right)) = \left(\frac{1+s}{2}\right)\left[1 - \left(\frac{1+t}{2}\right)\right]$$

$$= \frac{1}{4}(1+s)(1-t), \quad -1 \le s < t \le 1. \tag{3.3.21}$$

Now, by Lemma 3.3.3 and Theorem 3.3.3, $Q_n^*(t)$ converges in law to Q(t), where

$$Q(t) = Y^*(t) + Y^*(-t) - \int_0^1 \left[Y^*(w) + Y^*(-w)\right] dw, \quad 0 \le t \le 1. \tag{3.3.22}$$

In view of Lemma 3.3.4, it follows that Q(t) is a Gaussian process, which proves the lemma.

Lemma 3.3.6:

Let Q(t) denote the Gaussian process given by (3.3.22). Then

$$Cov(Q(s),Q(t)) = -t + \frac{1}{2}(s^2 + t^2) + \frac{1}{3}, \quad 0 \le s < t \le 1. \tag{3.3.23}$$

Proof:

By Theorem 3.3.2 and (3.3.20),

$$E(Q(t)) = E(Y^*(t)) + E(Y^*(-t)) - \int_0^1 \left[E(Y^*(w)) + E(Y^*(-w))\right] dw = 0$$

for $t \in [0,1]$. Hence, for $0 \le s < t \le 1$,

$$\text{Cov}(Q(s),Q(t)) = E\left\{\left[Y^*(s) + Y^*(-s) - \int_0^1 \left[Y^*(w) + Y^*(-w)\right] dw\right]\right.$$

$$\left.\times\left[Y^*(t) + Y^*(-t) - \int_0^1 \left[Y^*(w) + Y^*(-w)\right] dw\right]\right\}$$

$$= E(Y^*(s)Y^*(t)) + E(Y^*(s)Y^*(-t)) + E(Y^*(-s)Y^*(t))$$

$$+ E(Y^*(-s)Y^*(-t)) + E\left\{\left[\int_0^1 Y^*(w) + Y^*(-w)\ dw\right]^2\right\}.$$

$$- E\left\{\left[Y^*(s) + Y^*(-s) + Y^*(t) + Y^*(-t)\right]\int_0^1 \left[Y^*(w) + Y^*(-w)\right] dw\right\}.$$

Letting

$$J(t) = \int_0^1 E\left\{Y^*(t)\left[Y^*(w) + Y^*(-w)\right]\right\} dw$$

and using Theorem 3.3.2, the last expression becomes

$$K^*(s,t) + K^*(-s,t) + K^*(s,-t) + K^*(-s,-t)$$

$$- J(s) - J(-s) - J(t) - J(-t) + \int_0^1 \left[J(w) + J(-w)\right] dw. \qquad (3.3.24)$$

From (3.3.21),

$$J(t) = \int_0^1 \left[K^*(t,w) + K^*(t,-w)\right]\ dw$$

$$= \frac{1}{4}\left\{\int_0^t (1+w)(1-t)\ dw + \int_t^1 (1+t)(1-w)\ dw\right.$$

$$\left. + \int_0^1 (1-w)(1-t)\ dw\right\}$$

$$= \frac{1}{4}(1-t^2), \qquad (3.3.25)$$

for $t \in [0,1]$. Similarly,

$$J(-t) = \frac{1}{4}(1 - t^2). \qquad (3.3.26)$$

Thus,

$$\int_0^1 \left[J(w) + J(-w)\right] dw = \frac{1}{2}\int_0^1 (1-w^2)\, dw = \frac{1}{3}. \qquad (3.3.27)$$

Finally, substituting (3.3.21), (3.3.25), (3.3.26) and (3.3.27) in (3.3.24) gives

$$\mathrm{Cov}(Q(s),Q(t)) = -t + \frac{1}{2}(s^2 + t^2) + \frac{1}{3},\ 0 \le s < t \le 1,$$

as was to be shown.

Lemma 3.3.7:

The functional $z: D \to D$ defined by

$$z(x(t)) = x^2(t)$$

is continuous in the Skorokhod metric, d.

Proof:

Let $x(\cdot) \in D$ be fixed. It must then be shown that, given $\varepsilon > 0$, there exists an $\varepsilon' > 0$ such that for any $y \in D$, $d(x,y) < \varepsilon'$ implies that $d(x^2,y^2) \le \varepsilon$. Let $\varepsilon > 0$ be fixed, let

$$B = \max\left\{1, \sup_{0\le t\le 1} |x(t)|\right\}$$

and take $\varepsilon' = \varepsilon(2B + \varepsilon)^{-1}$. If $d(x,y) < \varepsilon'$, then, by the definition of d, given just before Theorem 3.3.3, there exists a $\lambda_o \in \Lambda$ such that

$$\sup_{0\le t\le 1} |t - \lambda_o(t)| \le \varepsilon(2B + \varepsilon)^{-1} < \varepsilon$$

and

$$\sup_{0\le t\le 1} |x(t) - y(\lambda_o(t))| \le \varepsilon(2B + \varepsilon)^{-1}.$$

Since

$$\sup_{0\le t\le 1} |x(t)+y(\lambda_o(t))| \le \sup_{0\le t\le 1} |x(t)| + \sup_{0\le t\le 1} |y(\lambda_o(t))| \le 2B+\varepsilon,$$

it follows that

$$\sup_{0\le t\le 1} |x^2(t)-y^2(\lambda_o(t))| \le \varepsilon.$$

In view of the definition of d, this implies that $d(x^2,y^2) \le \varepsilon$, which completes the proof of the lemma.

Theorem 3.3.4, giving the asymptotic null distribution of $\tilde{U}_n$, may now be stated and proved.

Theorem 3.3.4:

Let U be defined by

$$U = \int_0^1 Q^2(t)\, dt,$$

where Q(t) is a Gaussian process on [0,1] with

$$E(Q(t)) = 0$$

and

$$\mathrm{Cov}(Q(s),Q(t)) \equiv K(s,t) = -t + \frac{1}{2}(s^2+t^2) + \frac{1}{3}, \quad 0 \le s < t \le 1.$$

Then, for every $F \in \Phi_s$, $\tilde{U}_n$ converges in law to U as n tends to infinity.

Proof:

Since $U_n^{(\frac{1}{2})}$ is distribution free, without loss of generality F may be assumed to be the CDF of a uniform random variable on the interval [-1,1], that is,

$$F(t) = \begin{cases} 0 & \text{if } t < -1 \\ \frac{1}{2}(t+1) & \text{if } -1 \le t \le 1. \\ 1 & \text{if } t > 1 \end{cases}$$

Now, let

$$Q_n(t) = n^{\frac{1}{2}}\left[F_n(t) + F_n(-t) - 1 - \int_0^1 \left(F_n(w) + F_n(-w) - 1\right) dw\right], \quad -1 \le t \le 1.$$

Then,

$$Q_n(t) = Q_n(-t),$$

and

$$\tilde{U}_n = n \int_{-1}^{1} \left[F_n(t) + F_n(-t) - 1 - \int_{-1}^{1} (F_n(w) + F_n(-w) - 1) \frac{dw}{2}\right]^2 \frac{dt}{2}$$

$$= n \int_0^1 \left[F_n(t) + F_n(-t) - 1 - \int_0^1 (F_n(w) + F_n(-w) - 1)\, dw\right]^2 dt$$

$$= \int_0^1 Q_n^2(t)\, dt. \tag{3.3.28}$$

Since $\left(\frac{1+X}{2}\right)$ has a uniform distribution on [0,1] if X has a uniform distribution on [-1,1], it follows from Lemmas 3.3.5 and 3.3.6 that $Q_n(t)$ converges in law to a Gaussian process, Q(t), on [0,1] with mean zero and covariance kernel given by (3.3.23). Hence, by Theorem 3.3.3 and Lemmas 3.3.3 and 3.3.7, as n tends to infinity,

$$\int_0^1 Q_n^2(t)\, dt \to \int_0^1 Q^2(t)\, dt \quad \text{in law},$$

and, in view of (3.3.28), the proof is complete.

It has thus been shown that $\tilde{U}_n$, and hence, $U_n^{(a)}$, for any $a \in [0,1]$, converge in law to $U = \int_0^1 Q^2(t)\, dt$, where Q is a Gaussian process. The

problem now is to obtain the distribution of the random variable U. In what follows, it will be shown that the distribution of U is the same as the asymptotic null distribution of the well-known Cramér-von Mises goodness-of-fit statistic, ω_n^{gf} [7]. This result will follow from the fact that U can be expressed as a certain infinite sum involving independent χ^2 random variables. The following two theorems will be needed in establishing this representation for U.

Theorem 3.3.5 (Rosenblatt [14], pp. 185-95):

Let y(t) be a Gaussian process on [0,1] with continuous covariance kernel, K(s,t). Let λ_j and $f_j(\cdot)$, j=1,2,..., be the eigenvalues and corresponding eigenfunctions of the integral equation

$$\lambda f(t) = \int_0^1 K(s,t)f(s)\, ds, \tag{3.3.29}$$

satisfying the normalizing condition

$$\int_0^1 f_j(s)f_{j'}(s)\, ds = \delta_{jj'}, \tag{3.3.30}$$

where δ_{ij} denotes the Kronecker delta. Then the process

$$y_n(t) = \sum_{j=1}^{n} Z_j f_j(t)\lambda_j^{\frac{1}{2}},$$

where $\{Z_j;\ j=1,2,\ldots,n\}$ is a set of independent standard normal random variables, converges in mean square to the process y(t).

Theorem 3.3.6 (Kac and Siegert [9]):

Under the conditions of Theorem 3.3.5,

$$B = \int_0^1 y^2(t)\, dt = \sum_{j=1}^{\infty} \lambda_j Z_j^2 \quad \text{w.p. 1.}$$

The characteristic function of B, $\phi_B(\xi)$, is given by

$$\phi_B(\xi) = \prod_{j=1}^{\infty} (1 - 2i\ \xi\ \lambda_j)^{-\frac{1}{2}}.$$

An immediate consequence of Theorems 3.3.5 and 3.3.6 is that, with probability one,

$$U = \sum_{j=1}^{\infty} \lambda_j Z_j^2, \tag{3.3.31}$$

where $\{\lambda_j;\ j=1,2,\ldots\}$ is the set of solutions to the integral equation

$$\lambda f(t) = \int_0^1 \left[-\max\{s,t\} + \frac{1}{2}(s^2 + t^2) + \frac{1}{3}\right] f(s)\ ds \tag{3.3.32}$$

satisfying (3.3.30), and $\{Z_j^2;\ j=1,2,\ldots\}$ is a set of independent χ_1^2 random variables. Furthermore, the characteristic function of U is given by

$$\phi_U(\xi) = \prod_{j=1}^{\infty} (1 - 2i\ \xi\ \lambda_j)^{-\frac{1}{2}}. \tag{3.3.33}$$

Thus the distribution of U can be obtained from (3.3.33) if the solutions of (3.3.32) can be found. In this connection, note that since

$$\int_0^1 \left[-\max\{s,t\} + \frac{1}{2}(s^2 + t^2) + \frac{1}{3}\right] ds = 0,$$

$\lambda = 0$ and $f(s) \equiv 1$ give a solution to (3.3.32) satisfying (3.3.30). Any other eigenfunction, f, must therefore satisfy

$$\int_0^1 f(s)\ ds = 0. \tag{3.3.34}$$

Let f denote an eigenfunction corresponding to any non-zero eigenvalue, λ. Then

$$\lambda f(t) = \int_0^1 \left[-\max\{s,t\} + \frac{1}{2}s^2\right] f(s)\, ds + \left(\frac{1}{2}t^2 + \frac{1}{3}\right)\int_0^1 f(s)\, ds$$

$$= \int_0^t (-t)f(s)\, ds + \int_t^1 (-s)f(s)\, ds + \int_0^1 \frac{1}{2}s^2 f(s)\, ds.$$

Differentiating both sides with respect to t results in the equation,

$$\lambda f'(t) = (-t)f(t) - \int_0^t f(s)\, ds + tf(t) = \int_0^t f(s)\, ds.$$

Putting t=0 and t=1 on the right-hand side above and using (3.3.4), it follows that

$$f'(0) = f'(1) = 0. \tag{3.3.35}$$

Taking the derivative a second time shows that

$$\lambda f''(t) = -f(t).$$

That is,

$$\lambda f''(t) + f(t) = 0.$$

Now, the general solution of the differential equation above is given by (see, for example, [6], p. 506):

$$f(t) = \alpha \sin(\lambda^{-\frac{1}{2}}t) + \beta \cos(\lambda^{-\frac{1}{2}}t), \tag{3.3.36}$$

where α and β are constants to be determined using the boundary conditions, (3.3.30) and (3.3.35). The derivative of the general solution is

$$f'(t) = \lambda^{-\frac{1}{2}}\alpha \cos(\lambda^{-\frac{1}{2}}t) - \lambda^{-\frac{1}{2}}\beta \sin(\lambda^{-\frac{1}{2}}t).$$

Hence, by (3.3.35),

$$\lambda^{-\frac{1}{2}}\alpha \cos(0) - \lambda^{-\frac{1}{2}}\beta \sin(0) = 0,$$

from which it follows that $\alpha = 0$. Again from (3.3.35), with $\alpha = 0$,

$$\lambda^{-\frac{1}{2}}\beta \sin(\lambda^{-\frac{1}{2}}) = 0,$$

from which it follows that

$$\lambda^{-\frac{1}{2}} = j\pi, \quad j=1,2,\ldots .$$

Thus, the eigenvalues of (3.3.32) are given by

$$\lambda_j = (j\pi)^{-2}, \quad j=1,2,\ldots .$$

Finally, from (3.3.31),

$$U = \sum_{j=1}^{\infty} Z_j^2 (j\pi)^{-2} \quad \text{w.p. 1,}$$

so that from (3.3.33),

$$\phi_U(\xi) = \prod_{j=1}^{\infty} (1 - 2i\,\xi(j\pi)^{-2})^{-\frac{1}{2}}.$$

Durbin [7] has shown that under the null hypothesis the Cramér-von Mises goodness-of-fit statistic, ω_n^{gf}, converges in law to a random variable with characteristic function $\phi_U(\xi)$. Thus the asymptotic null distributions of ω_n^{gf} and $U_n^{(a)}$ are identical. Percentage points for this distribution have been tabulated by Anderson and Darling [1].

The result that ω_n^{gf} and $U_n^{(a)}$ have the same limiting distribution is interesting in view of the similar result that the Watson goodness-of-fit and two-sample statistics, μ_n^{gf} and μ_n^2 have the same asymptotic null distribution as the corresponding Kolmogorov-Smirnov two-sided statistics ([19] and [20]). No simple explanation has been found for either of these results.

3.4 The Exact Null Distribution of $U_n^{(\frac{1}{2})}$

As was the case in Chapter 2 for $T_n^{(a)}$, a closed form expression is not available for the exact null distribution of $U_n^{(a)}$. However,

using the method described in Section 2.3, exact critical values for $U_n^{(\frac{1}{2})}$ were calculated for various levels and sample sizes n = 9(1)20. These critical values are given in Table 3.

In order to compare the exact distribution of $U_{20}^{(\frac{1}{2})}$ with the asymptotic distribution, exact tail probabilities, P, for n = 15(1)20, were calculated using the critical values obtained from the asymptotic distribution. These probabilities are given in Table 4. Apparently, for sample sizes larger than 20, a test based on the α-level critical value of U will have an actual level less than α, with the difference between the actual level and α being less than the corresponding difference for n = 20, given in Table 4. This difference appears to decrease as n increases. A test using the α-level critical values for $U_{20}^{(\frac{1}{2})}$ when n is somewhat larger than 20 will likely have an actual level greater than, but closer to, α, than a test based on the asymptotic α-level critical values.

TABLE 3

EXACT CRITICAL VALUES FOR $U_n^{(\frac{1}{2})}$

$\alpha = .10$

n	x	$P(U_n^{(\frac{1}{2})} > x)$	x	$P(U_n^{(\frac{1}{2})} > x)$
9	0.29904	0.1016	0.30727	0.0859
10	0.29600	0.1094	0.30400	0.0977
11	0.31405	0.1006	0.31555	0.0986
12	0.31192	0.1011	0.31713	0.0991
13	0.31953	0.1008	0.32135	0.0999
14	0.32067	0.1007	0.32106	0.0997
15	0.32000	0.1003	0.32178	0.0998
16	0.32398	0.1016	0.32422	0.0999
17	0.32363	0.1004	0.32444	0.1000
18	0.32596	0.1001	0.32648	0.0992
19	0.32716	0.1003	0.32804	0.0999
20	0.32738	0.1003	0.32800	0.0995
∞	0.34730			

$\alpha = .05$

n	x	$P(U_n^{(\frac{1}{2})} > x)$	x	$P(U_n^{(\frac{1}{2})} > x)$
9	0.37037	0.0547	0.39506	0.0469
10	0.40000	0.0527	0.40100	0.0449
11	0.40421	0.0518	0.41473	0.0498
12	0.40741	0.0513	0.40914	0.0483
13	0.41875	0.0500	0.41966	0.0496
14	0.42310	0.0504	0.42456	0.0494
15	0.42193	0.0504	0.42370	0.0500
16	0.42554	0.0504	0.42749	0.0499
17	0.42825	0.0502	0.42866	0.0497
18	0.43090	0.0500	0.43210	0.0494
19	0.43126	0.0502	0.43155	0.0500
20	0.43200	0.0501	0.43238	0.0497
∞	0.46136			

TABLE 3 (Continued)

$\alpha = .01$

n	x	$P(U_n^{(\frac{1}{2})} > x)$	x	$P(U_n^{(\frac{1}{2})} > x)$
9	0.56790	0.0117	0.65295	0.0039
10	0.56900	0.0117	0.61600	0.0078
11	0.59654	0.0117	0.61758	0.0098
12	0.60359	0.0103	0.63194	0.0098
13	0.64087	0.0100	0.64907	0.0095
14	0.64140	0.0104	0.64723	0.0099
15	0.65600	0.0100	0.65659	0.0099
16	0.65625	0.0102	0.65991	0.0100
17	0.66436	0.0100	0.66517	0.0099
18	0.67164	0.0101	0.67215	0.0100
19	0.67298	0.0100	0.67357	0.0099
20	0.67738	0.0101	0.67800	0.0100
∞	0.74346			

$\alpha = .001$

n	x	$P(U_n^{(\frac{1}{2})} > x)$	x	$P(U_n^{(\frac{1}{2})} > x)$
9	0.65295	0.0039	0.74074	0.0000
10	0.74400	0.0020	0.82500	0.0000
11	0.76033	0.0029	0.83396	0.0010
12	0.85417	0.0015	0.92303	0.0005
13	0.85662	0.0015	0.90123	0.0010
14	0.90671	0.0012	0.93477	0.0010
15	0.93333	0.0011	0.94637	0.0010
16	0.95679	0.0010	0.97241	0.0009
17	0.97619	0.0010	0.97659	0.0010
18	0.99263	0.0010	0.99314	0.0010
19	0.99898	0.0010	0.99927	0.0010
20	1.00638	0.0010	1.00738	0.0010
∞	1.16786			

TABLE 4

EXACT TAIL PROBABILITIES FOR $U_n^{(\frac{1}{2})}$ BASED ON THE α-LEVEL CRITICAL VALUES OF U

$$P = P(U_n^{(\frac{1}{2})} > U_\alpha)$$

	α = .10		α = .05	
n	P	α − P	P	α − P
15	0.0837	0.0163	0.0382	0.0118
16	0.0848	0.0152	0.0395	0.0105
17	0.0858	0.0142	0.0401	0.0099
18	0.0858	0.0142	0.0411	0.0089
19	0.0873	0.0127	0.0413	0.0087
20	0.0884	0.0116	0.0419	0.0081

	α = .01		α = .001	
n	P	α − P	P	α − P
15	0.0055	0.0045	0.0002	0.0008
16	0.0056	0.0044	0.0002	0.0008
17	0.0059	0.0041	0.0002	0.0008
18	0.0060	0.0040	0.0002	0.0008
19	0.0064	0.0036	0.0003	0.0007
20	0.0065	0.0035	0.0003	0.0007

CHAPTER 4

PROBLEMS FOR FURTHER RESEARCH

Many problems in the area of symmetry tests remain to be solved. As mentioned above, the asymptotic null distribution of the statistics in the T_n class is not known. Also, it would be desirable to obtain a method for determining the power of the tests based on statistics in any of the four classes discussed above against various types of alternatives. In addition to the Cramér-von Mises type symmetry statistics, there exist corresponding EDF symmetry statistics based on the Kolmogorov-Smirnov statistic ([3], [4], [11] and [16]). The Bahadur efficiencies of several of these statistics, relative to each other, have been obtained by Littell [11], but there are no criteria available for comparing the Cramér-von Mises with the Kolmogorov-Smirnov statistics. The two invariance properties discussed in previous chapters, as well as the two comparisons upon which the Cramér-von Mises type statistics are based, have not yet been investigated with respect to the Kolmogorov-Smirnov type symmetry statistics.

In the area of distribution theory, it has been shown above that $U_n^{(a)}$ and the Cramér-von Mises goodness-of-fit statistic have the same asymptotic null distribution. Also, as noted in Section 3.3, the Watson goodness-of-fit and two-sample statistics have the same asymptotic null distribution as the corresponding two-sided Kolmogorov-Smirnov

statistics. As it appears that these two distributions are common ones, it might be possible to determine when a particular statistic will have one of them as its asymptotic distribution.

Finally, in many practical situations, the center of symmetry, θ, may not be known. It would thus be desirable to find a distribution free EDF test for symmetry when θ is estimated. Since most distribution free tests lose this property when parameters are estimated, this problem may be impossible to solve for small samples, but asymptotically distribution free procedures may possibly be obtained.

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BIOGRAPHICAL SKETCH

David Lawrence Hill was born on July 11, 1949, in Morristown, New Jersey, and raised in nearby Bernardsville. Upon graduation from Bernards High School in June, 1967, he attended Bucknell University in Lewisburg, Pennsylvania, and received the degree of Bachelor of Science in Mathematics in May, 1971. While at Bucknell, he became interested in statistics through the influence of the late Professor Paul Benson. In September, 1971, David enrolled in the graduate school at the University of Florida, and, in June, 1973, was awarded the degree of Master of Statistics. Since then, he has been working toward the degree of Doctor of Philosophy, and has been the recipient of a graduate fellowship and a research assistantship from the University of Florida.

After receiving his degree, David will assume a position as an Assistant Professor in the Department of Mathematical Sciences at Northern Illinois University in DeKalb, Illinois. He is a member of the American Statistical Association.

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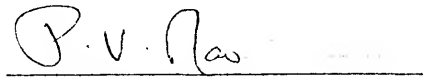

P. V. Rao, Chairman
Professor of Statistics

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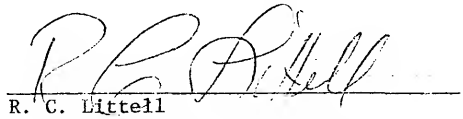

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D. D. Wackerly
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This dissertation was submitted to the Graduate Faculty of the Department of Statistics in the College of Arts and Sciences and to the Graduate Council, and was accepted as partial fulfillment of the requirements for the degree of Doctor of Philosophy.

August, 1976

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